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The Vertex as a Bogoliubov Transformed Vacuum State in String Field Theory

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Abstract

We develop the point of view that the string vertex is nothing but the Bogoliubov-transformed $SL(2,R)$-invariant out-ground state induced by the conformal (Mandelstam) mapping. We explicitly demonstrate this idea, using the open string field theory of Witten. The notion of asymptotic string fields is introduced and the vertex is constructed without referring to the conventional delta-function overlap equation. The ghost insertion factors are understood as a consequence of the presence of the kernels in the ghost Bogoliubov transformations. We prove the vanishing of the anomaly of the BRST charge under the Bogoliubov transformations at $d = 26$, from which the BRST invariance of the vertex associated with any Mandelstam map follows immediately. Underlying physical pictures are contemplated.

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I. Introduction

The gauge invariant covariant functional field theory of string has presently received much attention[1]. It has been hoped that it will offer a nonperturbative framework for the problems which we cannot address intelligibly in local field theory based on point particles. A crucial role is played by the BRST formulation of the first-quantized string[2]. The Fock space consisting of single string states provides a set of free field bases for the string field functional, giving a precise meaning to the rather fragile notion of functional fields. The expansion using these bases implies representation of a free string field in terms of infinite component local fields: the lowest few terms correspond to the particles we observe in nature.

A novel approach has been taken by Witten[3]. Here, string fields are postulated to be semi-infinite Lie-algebra-valued differential forms on the Virasoro group. Operations such as multiplication or integration are introduced through axioms which ordinary Lie algebra valued differential forms obey.

In any of the approaches proposed so far, what is central in the formulation is the construction of the vertex – a first quantized state which tells us how strings join together. A primary ingredient is the traditional delta-function overlap equation. For the sake of our discussion, let us briefly recapitulate the historical route leading to this. The conformal mapping invented by Mandelstam[4] in the first quantized lightcone formulation played an essential role in exhibiting the vertex appearing in dual amplitudes as an interaction of strings. Soon after, Kaku and Kikkawa[5] and also Cremmer and Gervais[6] realized that the Neumann function vertex in Mandelstam's picture is nothing but a representation of (infinite dimensional) delta function interactions. This led them to the lightcone interacting string field theory. The construction of interactions in the covariant string field theory has since progressed[7], taking the form of inheriting those tools which were developed in the lightcone framework[8]. Conformal maps relevant to Witten string field theory have been found[9,10,11,12].

All these formulations would be satisfactory and probably sufficient as long as one's purpose is tied to the classical perturbation expansion performed in the interaction picture. There is, however, one essential aspect of the string which the traditional approach, namely the free field representation combined with the delta function interaction, can hardly illuminate; namely, string interactions take place without specifying where they occur.

In this paper, we are going to develop a new framework for the construction of vertices. The construction is made without recourse to the traditional delta-function overlap equation mentioned above, providing an independent and alternative framework. The notion of asymptotic fields familiar to us in local field theory in the Heisenberg picture carries through in our point of view.

A central claim in our approach is the existence of a set of bases for string fields in which the vertex is represented as a ground state of single string oscillators. The connection to the conventional form based on the free field bases is made through the Bogoliubov transformation specified unambiguously by the Mandelstam map.

The existence of such bases is not a surprise; rather, it is naturally suggested by the very existence of the Mandelstam map as a tool for evaluating the Neumann coefficients of the scattering geometry in question. Let us, for definiteness, consider a conformal map from some region of the unit disk in the z plane to the upper half part of the unit disk in w_r plane ($r = 0, 1, \cdots, N-1$). In the $\log w_r$ plane, we see a semi-infinite strip with width π which represents a surface swept by a string. In the z plane, a single string coordinate is expanded by "universal bases" $\{\{z^n;\ n \in \mathcal{Z}\}\}$. The same object, after the map, is now expanded in the w_r ($r = 0, \cdots, N-1$) planes by "many string bases" $\{\{w_r^n;\ r = 0, \cdots, N-1,\ n \in \mathcal{Z}\}\}$. The change of bases from the universal bases to the many string bases induces a Bogoliubov transformation between the corresponding two sets of oscillators. The ground state defined in the universal bases is found to be a superposition of excited states if one sees things in the many string bases. A situation of this kind is familiar to us in the study of quantum field theory in curved spacetime[13].

Once one includes the ghost and anti-ghost coordinates in question, our claim amounts to the statement that the vertex is an $SL(2,R)$- invariant ground state in the universal bases. The standard expression of the vertex written using the Neumann coefficients is obtained by expressing the ground state in terms of many string bases.

It has been observed that the ghost part of the vertex in general requires insertion factors in addition to the ordinary exponential of quadratic forms. We understand these as the kernels of the ghost Bogoliubov coefficients. We will give explicit examples. An analogous result has been known by Christ in his study of axial vector anomalies[14]. The relation to the Riemann-Roch index theorem or ghost number anomaly is now manifest.

Our claim provides a fresh and elegant point of view for BRST invariance of the vertices. The BRST charge is, by definition, normal ordered with respect to the ground state of the many-string bases. The BRST invariance of the vertices translates therefore into the statement that the charge is again normal ordered with respect to the ground state in the universal bases. This in turn means that the BRST charge must not be anomalous under the Bogoliubov transformation induced by the conformal map: $z \mapsto w_r(z)$.

We shall explicitly calculate the anomaly and show

$$Q_{BRS} - :Q_{BRS}:^{out} = -\oint \frac{dz}{2\pi i} c^z(z) \left(\frac{d-26}{12}\right)\left(\frac{\dddot{w}}{\dot{w}} - \frac{3}{2}\frac{\ddot{w}^2}{\dot{w}^2}\right) \qquad (*) \ .$$

Any vertex constructed unambiguously from the Mandelstam map is guaranteed to be BRST invariant in $d = 26$. No further proof is necessary. Whether it obeys the overlap equations is a separate question. The BRST invariance of the vertices is a genetic property of the charge itself rather than the one depending on the form of the vertices. The appearance of the Schwarzian derivative is reasonable from the transformation property of the stress-energy tensor.

Contrary to the impression one might get at first, the construction of the Bogoliubov coefficients from the Mandelstam map is a subtle problem because of the singularities of the Mandelstam map. They represent a string - an extended object - identified as a point at infinity in the $\log w_r$ plane. We will see that the Bogoliubov coefficients contain contributions from contour integrals around these "punctures".

A heuristic physical picture underlying our formalism might be the following. The splitting and joining of strings gives rise to a change of the induced metric of the two dimensional world sheet. Oscillators living in the two dimensional spacetime will feel this change of the background gravitation and the creation and annihilation operators are inevitably mixed. We must, however, caution you by saying that, in the BRST formalism we will follow, one is not allowed to introduce world sheet metric as degrees of freedom. The above pictorial argument is replaced ,therefore, by a formal normal ordering problem.

Most discussion of this paper, in particular, the result $(*)$ and the insertion factors as kernels of Bogoliubov coefficients are valid to any open string field theory one has in mind and presumably to a full-fledged closed string field theory one hopes to have. For the sake of an explicit demonstration of our claim, we use, however, the vertices relevant to the Witten open string field theory. These include the fundamental three point vertex and its N-point generalization. The application of our Bogoliubov transformation approach to Neven-Schwarz type vertices[15] will be given elsewhere. We also plan to report on the corresponding examination for joining-splitting vertices together with remaining points[16].

In Section 2, we formulate interacting string field theory, using the notion of asymptotic fields familiar from local field theory. We state the main claim of our work. A connection with the traditional δ-function overlap equations is made. In Section 3, we briefly review the conformal (Mandelstam) mapping for the type of vertices mentioned above. The construction of the Bogoliubov coefficients is given in Section 4 for the bosonic part. In Section 5, we extend this to the ghost part of the vertex. In Section 6, we continue and complete the discussion of Section 2 , demonstrating the claim explicitly in the operator form. In section 7, we discuss in detail the problem of ghost insertion factors, exploring the kernel structrue of the ghost Bogoliubov coefficients, establishing the ghost number counting in our framework. The BRST invariance is proved [1], [2] from the above universal point of view in Section 8. The final section is devoted to discussion and outlook. The proofs of several mathematical properties used in the text are given in Appendix A, B, C and D.

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[1] We should mention that the use of the universal z bases has been considered in the conformal field theory approach in ref.(17) and ref.(18). Here we take an explicitly time-dependent point of view without relying upon the connectivity conditions.

[2] While the current approach was put forward by us, Kugo, Kunitomo and Suehiro wrote a preprint (ref.(19)) on the improved proof of the BRST invariance of the vertex of ref.(8). As they admit in their preprint, the basic idea is based on our idea which one of us informed at the workshop in Osaka, although some come from their early work. Their proof is still specific to the theory of ref.(8) since they use the explicit form of the connectivity conditions. The explicit use of the connectivity condition prevented them from revealing the general geometrical property underlying the BRST invarianve of the vertices. We will make more comments on these in Section 8.

II. The Vertex As The Out-Ground State

In this section, we begin by reviewing briefly the conventional notion of vertices based on overlap equations tied with the free string field, namely the string field in the interaction picture. We then reinterpret it as an incoming asymptotic string field, which can be regarded as a Heisenberg operator after second quantization. We develop a general framework for the construction of the vertices based on the Bogoliubov transformation, namely mode mixing between two sets of oscillators serving as bases of the string field at remote past and remote future. We have a general gauge invariant open string covariant field theory in mind in most of the discussions.

The main purpose of this section is to demonstrate the independence of our framework from the traditional one based on the delta-function overlap. We will see that the vertex constructed below is more general and may describe situations not obeying the delta-function overlap.

Let us, for a moment, ignore the ghost degrees of freedom and consider bosonic ones only. We shall deal with the integration of a product of N string fields:

$$\int \Psi_1 * \Psi_2 * \cdots \Psi_N \ . \tag{2.1}$$

Here, Ψ_i is a functional of the momentum density of a string at a particular time slice. The conventional definition of the above quantity in the interaction picture, namely in the free field representation, is expressed as

$$\int \prod_{r=0}^{N-1} \mathcal{D}p_r(\sigma)(\prod_{r=0}^{N-1} D_r \Psi^{(0)}[\, p_r(\sigma)]) = \left\langle V_N^{(0)} \mid \Psi^{(0)} \right\rangle_1 \cdots \mid \Psi^{(0)} \Big\rangle_N \ . \tag{2.2}$$

The superscripts indicate the particular time slice chosen. Here the factor D_r inserted is an (infinite) product of delta functions

$$D_r = \prod_{\sigma \in d_r} \delta(f_r(\sigma)) \ . \tag{2.3}$$

and the arguments $f_r(\sigma)$ of the delta functions are called overlaps. The one chosen for Witten string field theory is

$$f_r(\sigma) = p_r(\sigma) + p_{r+1}(\pi - \sigma) \ , \ \ \sigma \in d_r = [0, \pi/2] \ . \tag{2.4}$$

The insertion factor eq. (2.3) ensures, at d_r, the local conservation of the momentum density of N string surfaces which must form a single Riemann surface. (See Fig. 1.) In eq. (2.2), we also represented the string field as a ket vector of a single string Hilbert space set up at the time slice. The bra vector $\left\langle V_N^{(0)} \right|$ which sends a direct product of ket vectors into a complex number is what we call vertex. In principle, eq. (2.2) can be used to define $\left\langle V_N^{(0)} \right|$, but in practice it is defined by imposing the overlap conditions:

$$\left\langle V_N^{(0)} \right| f_r(\sigma) = 0, \qquad r = 0, \cdots N-1, \;\; \sigma \in d_r \;\; . \tag{2.5}$$

A few remarks are worth mentioning. The momentum density at an arbitrary point (τ_r, σ_r) of the rth string is written as

$$p(w_r) = \sum_n \alpha_n^r w_r^n, \;\; \text{with} \;\; w_r = e^{\tau_r + i\sigma_r} \;\; . \tag{2.6}$$

In the covariant formulation, there is no notion of a time universally defined over a single string. One has to introduce infinitely many times $t_r(\sigma)$. String fields are, in general, functionals of $p(w_r)$ at a particular time slice expressed as conditions among $t_r(\sigma)$ and τ_r:

$$\Psi[\, p(w_r);\; t_r(\sigma) = \text{fixed}, \tau_r = \text{fixed}] \;\; . \tag{2.7}$$

Obviously, the time slice in eq. (2.2) corresponds to

$$t_r(\sigma) = \tau_r = 0 \;\; , \tag{2.8}$$

and

$$\Psi^{(0)}[\, p(\sigma)\,] \equiv \Psi[\, p(w_r)\,;\; t_r(\sigma) = \tau_r = 0\,] \;\; . \tag{2.9}$$

The ket vectors at another time slice in the interaction picture is related to the previous ones by the evolution operator since they are the same state vectors. There is, however, one physically distinct choice among various time slices: it is $t_r(\sigma) = \tau_r = -\infty$. Eq. (2.2) is now written as

$$\left\langle V_N^{in} \,\middle|\, \Psi^{in} \right\rangle_1 \cdots \left| \Psi^{in} \right\rangle_N \;\; . \tag{2.10}$$

The vector $\left\langle V_N^{in} \right|$ satisfies the same overlap equation :

$$\left\langle V_N^{in} \right| f_r^{in}(\sigma) = 0 \quad . \tag{2.11}$$

Here, $f_r^{in}(\sigma)$ is obtained from eq. (2.4) by replacing $p_r(\sigma)$ by $p^{in}(w_r) = \lim_{t_r \to -\infty} p(w_r)$.

So far, the discussion has been about free string fields and two sets of bases set up at two different time slices are trivially related. Let us imagine, for a while, working on general curved surfaces. The evolution of string configuration in intermediate times is subject to modifications. But one can invoke the conventional asymptotic conditions to keep all string configurations at $t(\sigma) = -\infty$ incoming asymptotic free configurations. In other words, a Fock space built out of the direct product of N "in-ground state" $\times_{r=0}^{N-1} \left| 0\ in \right\rangle_r$, by acting a set of oscillators $\{\{\alpha_n^r;\ \ r = 0, 1, \cdots, N - 1,\ n \in \mathcal{Z}\}\}$, is providing a set of bases at $t(\sigma) = -\infty$ for interacting string fields, namely string fields in the Heisenberg picture after the second quantization.

For the sake of later discussions, we introduce the following coordinates for incoming string

$$\begin{aligned} \alpha_{w_r}(w_r) &= \sum_n \alpha_n^{r\ in} w_r^{-n-1}, \\ &= \frac{1}{w_r} p^{in}(w_r) \ \ , r = 0, \cdots N-1,\ w_r \in \mathcal{R}_r \end{aligned} \tag{2.12}$$

We indicate, by the subscript, that $\alpha_{w_r}(w_r)$ is a weight one conformal field in the first quantized language. It is expanded by the Laurent series with a set of bases $\{\{w_r^n;\ r = 0, \cdots N-1\ n \in \mathcal{Z}\}\}$. We call these bases for the first quantized operators the many-string bases. The operator $\alpha_{w_r}(w_r)$ is originally defined in the upper half part of the unit disk $\mathcal{R}_r$. The open-string boundary conditions require that it be extended to the lower half part analytically.

Let us first draw an alternative pictorial view to Fig. 1 as this turns out to be the basis of our intuitive understanding for what we develop later. (See Fig. 2.) We prepare, at $t_r(\sigma) = -\infty$, incoming string configurations carrying definite local momentum densities. The in-bases are already given. A natural question arises: what are the asymptotic out-bases? From the figure, we see the strings overlapped with each other to become a single string. We therefore postulate the existence of a single string Fock space as asymptotic out-bases. The relevant coordinate with conformal weight one is

$$\alpha_z(z) = \sum_n \alpha_n^{out} z^{-n-1}, \quad z \in \mathcal{D} \ . \tag{2.13}$$

with $\mathcal{D}$ being a domain in the complex plane. We call the bases $\{\{z^n;\ n \in \mathcal{Z}\}\}$ for the first quantized operator universal string bases. We often suppress the superscripts "in" and "out" from now on. The two objects $\alpha_{w_r}(w_r)\ \ r = 0, \cdots N-1$ and $\alpha_z(z)$ are related by the transformations given by the evolution operator. On the other hand, they must be transformed as conformal fields. We therefore conclude

$$\alpha_z(z) = U_r \alpha_{w_r}(w_r) U_r^{-1} = \frac{dw_r}{dz}\alpha_{w_r}(w_r), \ \ \text{for } \ z \in \mathcal{D}_r \ . \tag{2.14}$$

Here, $\mathcal{D}_r$ is a segment of the domain $\mathcal{D}$ and $\mathcal{D} = \bigcup_{r=0}^{N-1}\mathcal{D}_r$. Alternatively, one can write

$$\alpha_z(z) = \sum_{r=0}^{N-1} \frac{dw_r(z)}{dz}\alpha_{w_r}(w_r)\theta_r, \ \ \text{with } \ \theta_r = \begin{cases} 1 & \text{if } z \in \mathcal{D}_r \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

The map has been known generically as a Mandelstam map. It has been used as a technique for evaluating Neumann coefficients appearing in the expression of the vertices. The pictorial view represented by Fig. 2 and its relation to the map seem to have been unclear. We, in a way, promoted the notion of the map to a statement in the Hilbert space of the first quantized string.

On what grounds, can we argue for our postulated out-bases? Let us, for that purpose, work with the special (and traditional) case where two neighbouring domains $\mathcal{D}_r$ and $\mathcal{D}_{r'}$ are intersected with a line. It is easy to see that the existence of a single-string Fock space as out-bases for string fields already implies the overlap condition (2.11) by a judicious choice of the Mandelstam map. Let z be located at an intersection of $\mathcal{D}_r$ and $\mathcal{D}_{r'}$. Since $\alpha_z(z)$ is regular around this point, we have, for arbitrary state $\mid f \Big\rangle$ in the many-string Fock space,

$$\Big\langle V \mid \frac{dw_r}{dz}\alpha_{w_r}(w_r) - \frac{dw_{r'}}{dz}\alpha_{w_{r'}}(w_{r'}) \mid f \Big\rangle = 0 \ . \tag{2.16}$$

Choosing the map $\frac{dw_r}{dw_{r'}} = -\frac{w_r}{w_{r'}}$, we see that the overlapping condition (2.11) is satisfied. (See a more precise argument we make later.)

So far, all the discussions have been about bosonic string degrees of freedom. It is straightforward to extend them to the ghost degrees of freedom. We employ the original fermionic description of the ghosts and the antighosts. The relevant first quantized operators for the asymptotic in-bases for the ghost sector are

$$c^{w_r}(w_r) = \sum_n c_n^{(r)} w_r^{-n+1} \tag{2.17}$$

$$b_{w_r w_r}(w_r) = \sum_n b_n^{(r)} w_r^{-n-2} \quad r = 0, \cdots, N-1 \ . \tag{2.18}$$

They are ghost and antighost coordinates which have conformal weights minus one and two respectively. The in-bases are obtained by acting the creation operators on the product of the ground state of each string, which we define to be $|\uparrow>_r$: namely

$$b_n^{(r)} | \uparrow>_r = 0 \ , \quad n \geq 1 \tag{2.19}$$

$$c_n^{(r)} | \uparrow>_r = 0 \ , \quad n \geq 0 \ . \tag{2.20}$$

Following eq. (2.13), we introduce

$$b_{zz}(z) = \sum_n b_n z^{-n-2} \text{ and} \tag{2.21}$$

$$c^z(z) = \sum_n c_n z^{-n+1} \ . \tag{2.22}$$

Their single-string Fock space is providing out-bases for the ghost part of string fields. Eqs. (2.18), (2.19) and eqs. (2.20), (2.21) are related by

$$c^z(z) = \sum_r \frac{dz}{dw_r} c^{w_r}(w_r)\theta_r \ , \text{ and} \tag{2.23}$$

$$b_{zz}(z) = \sum_r \left(\frac{dw_r}{dz}\right)^2 b_{w_r w_r}(w_r)\theta_r \ . \tag{2.24}$$

The overlapping condition analogous to eq. (2.11) are automatically implied by the existence of the operators (eqs. (2.22), (2,23)), living on the complex z plane as in eq. (2.16).

We have seen that the existence of the out-bases consisting of a single string Fock space implies the traditional overlap equation when the map is appropriately chosen. Reproducing the overlap eq. (2.16) is, however, not the direction we put forward in the rest of the paper. As is mentioned before, we rather develop a framework for the construction of the vertices which is just based on the existence of the in- and out-bases. The following claim is an integral part of our formalism.

<u>Claim</u>

The vertex appearing in string field theory is an $SL(2,R)$ invariant out- ground state, that is, it satisfies

$$\begin{aligned} \Big\langle V \mid \alpha_n &= 0 \quad , n \le 0 \ , \\ \Big\langle V \mid b_n &= 0 \quad , n \le 1 \ , \\ \Big\langle V \mid c_n &= 0 \quad , n \le -2 \ . \end{aligned} \tag{2.25}$$

We will prove this explicitly in the operator form in later sections, taking the vertex of Witten string field theory and its N point generalizations. Let us give here a less rigorous argument to get a feeling about why out-annihilation oscillators must annihilate the vertex. The above claim implies that, for any final eigenstate $\Big\langle f \ ; \ t(\sigma) = t \to +\infty \mid$ in the many-string Fock space, $\Big\langle f, \ t(\sigma) = t \to +\infty \mid V^{in}, \ t(\sigma) = -\infty \Big\rangle = 0$ unless f is a ground state. This can be readily seen from the expression

$$\lim_{it\to\infty} \sum_n \Big\langle f|n, t_0 \Big\rangle \exp(-i\epsilon_n(t-t_0)) \Big\langle n, t_0 | V^{in}, t(\sigma) = -\infty \Big\rangle \tag{2.26}$$

by invoking the conventional continuation to the imaginary time axis and the asymptotic condition. The time t_o is the time where the surface becomes flat. The large time propagation after the interaction projects the state into the final ground state.

The above argument is, however, not completely pleasing : in the covariant formulation, a negative metric for string oscillators is present and therefore the continuation procedure should be more subtle. A somewhat related objection about the claim is the following: the claim implies, for an arbitary state f, $\Big\langle V \mid \alpha_z(z), b(z), c(z) \mid f \Big\rangle$

is regular as $z \to 0$ *i.e.* at the interaction point. But in the conventional approach, this point has been found to be singular and the vertex does not obey the overlap condition in general. We shall deal with this paradox by "regulating" the conformal map (Section 3). The overlap condition with the regulator is no longer a delta function type. We will understand insertion factors in a more general way in Section 7.

In the rest of this section, we will assume the claim and set up our framework without using eqs. (2.11) and (2.25). From eqs. (2.15), (2.23) and (2.24) alone, we see that in-oscillators and out-oscillators are linearly related. We spell out these Bogoliubov transformations to establish a notation;

$$\begin{aligned} \alpha_n &= \sum_{\ell,r} A_n^{(r)\ell}\alpha_\ell^{(r)} \quad , \quad \alpha_\ell^{(r)} = \sum_n \Lambda_\ell^{(r)n}\alpha_n \\ c_n &= \sum_{\ell,r} P_n^{(r)\ell}c_\ell^{(r)} \quad , \quad c_\ell^{(r)} = \sum_n R_\ell^{(r)n}c_n \\ b_n &= \sum_{\ell,r} Q_n^{(r)\ell}b_\ell^{(r)} \quad , \quad b_\ell^{(r)} = \sum_n S_\ell^{(r)n}b_n \ . \end{aligned} \tag{2.27}$$

We call $A_n^{(r)\ell}$, $P_n^{(r)\ell}$ and $Q_n^{(r)\ell}$ Bogoliubov coefficients and $\Lambda_\ell^{(r)n}, R_\ell^{(r)n}$ and $S_\ell^{(r)n}$ inverse Bogoliubov coefficients respectively. We are going to evaluate them directly from the Mandelstam map in Section 4 and 5. We will also show that the transformation must be understood in terms of certain matrix elements. Let us take here that these coefficients are given and express the vertex state in terms of in-bases.

Write the vertex state $\langle V_N \mid$ by a direct product of the state ${}^X\langle V_N \mid$ for the bosonic oscillators and ${}^{gh}\langle V_N \mid$ for the (anti-) ghosts: namely,

$$\langle V_N \mid = {}^X\langle V_N \mid \times {}^{gh}\langle V_N | \ . \tag{2.28}$$

First consider the bosonic part ${}^X\langle V_N \mid$, which satisfies, according to the claim,

$${}^X\langle V_N \mid \alpha_n = {}^X\langle V_N \mid \sum_{s=0}^{N-1}\sum_m A_n^{(s)m}\alpha_m^{(s)} = 0 \ , \quad n \leq 0 \ . \tag{2.29}$$

The solution must be the exponential of the quadratic form :

$$ {}^{X}\Big\langle V_N \mid = \times_{s'=0}^{N-1} \; {}_{s'}\Big\langle 0 \mid \exp\left[-\frac{1}{2}\sum_{r,s=0}^{N-1}\sum_{\ell,m=0}^{\infty}\alpha_\ell^{(r)} N_{\ell m}^{rs}\alpha_m^{(s)}\right] \;, $$

$$ \text{with} \qquad {}_{s}\Big\langle 0 \mid \alpha_\ell^{(r)} = 0, \quad \ell \leq -1 \;. \tag{2.30} $$

Here, $\alpha_0^{(r)} \equiv p^{(r)} = a_0^{(r)} + a_0^{(r)\dagger}$, $x^{(r)} = \frac{i}{2}(a_0^{(r)} - a_0^{(r)\dagger})$ and $[a_0^{(r)}, a_0^{(r)\dagger}] = 1$. Also, $\Big\langle 0 \mid a_0^{(r)\dagger} = 0$, *i.e.* it is the oscillator-ground state.

The coefficient $N_{\ell m}^{rs}$ must satisfy

$$ A_n^{(r)m} - \sum_{s=0}^{N-1}\sum_{\ell=1}^{\infty} A_n^{(s)-\ell}\ell N_{\ell m}^{rs} = 0, \quad n \leq 0 \;, $$

$$ \text{for } m \geq 0 \text{ and } \; r = 0, 1 \cdots, N-1 \;. \tag{2.31} $$

As we shall show the details in Appendix A, the solution to eq. (2.31) is unique and given by the standard Neumann coefficients.

To identify $N_{\ell m}^{rs}$ in our framework, we calculate the following two point function in two different ways:

$$ {}^{X}\Big\langle V_N \mid \alpha_{-\ell}^{(r)}\alpha_{-m}^{(s)} \mid \bar{0}\Big\rangle \;,\text{for } \; \ell, m \geq 1 \;, \quad \alpha_\ell^{(s)} \mid \bar{0}\Big\rangle = 0 \;, \ell \geq 0 \;. \tag{2.32} $$

Here, $\mid \bar{0}\Big\rangle$ denotes the zero-momentum ground state. Using in-oscillators, one finds that it equals $\ell m N_{\ell m}^{rs}$. One can also evaluate this quantity, converting to out-oscillators, using inverse Bogoliubov coefficients. We obtain, in this way, a formula

$$ \ell m N_{\ell m}^{rs} = \sum_{n\geq 1}\Lambda_{-\ell}^{(r)n} n \Lambda_{-m}^{(s)-n} + \sum_{n\geq 1,\, n'\geq 1}\Lambda_{-\ell}^{(r)n}\Lambda_{-m}^{(s)n'} \frac{{}^{X}\Big\langle V \mid \alpha_n\alpha_{n'} \mid \bar{0}\Big\rangle}{{}^{X}\Big\langle V \mid \bar{0}\Big\rangle} \;. \tag{2.33} $$

As we shall show later, the contribution from the second term is zero. The case $\ell = 0$ or $m = 0$ is defined by the limit of the above equation. Eq. (2.33), therefore, identifies $N_{\ell m}$ in terms of inverse Bogoliubov coefficients.

The ghost part of the vertex can be worked out in a similar way. But we defer it to later sections since the uniqueness of the ghost counterpart of eq.(2.31) does not hold any longer. We will see that it is precisely the origin of the ghost insertion factors.

The above discussion of the construction of vertices essentially demonstrates the independence of our framework from the traditional one based on the delta-function overlap. We finish this section by proposing the following procedure for constructing vertices. Prepare in-bases consisting of many-string Fock spaces and out-bases consisting of a single-string Fock space. The vertex is, then, defined by specifying the Mandelstam map and demanding the claim. The expression in terms of in-oscillators is given by eq. (2.30). (See, eq. (7.15) for the ghost part.) The vertex constructed this way is clearly more general than the ones based on traditional framework. We can construct vertices from the Mandelstam map which does not quite yield the ordinary overlap equation. Of course,the question whether these general vertices satisfy other criteria, is a separate issue. In Section 8, we shall show the BRST invariance of the general vertices. For $N = 3$, it implies an order g gauge invariance for the string field theory having the action of Chern-Simons form.

III. Conformal Mapping

Let us briefly recapitulate the conformal mapping for the N-vertex introduced in ref.(9). The universal z-plane is decomposed into N domains in such a way that

$$\{ \ z - \text{plane} \ \} = \bigcup_{s=0}^{N-1} \mathcal{D}_s,$$

$$\mathcal{D}_s = \{ z \mid \frac{2\pi}{N} s \leq \arg z \leq \frac{2\pi}{N}(s+1) \}, \qquad s = 0, 1, \cdots, N-1 \ . \tag{3.1}$$

The $s-$th string lives in the domain $\mathcal{D}_s$ defined above and its coordinates are given by the conformal mapping

$$\tau_s + i\sigma_s \ = \ log w_s \ , \tag{3.2}$$

$$
\begin{aligned}
w_s(z) &= -i\frac{z^{N/2}-ie^{-\epsilon}}{z^{N/2}e^{-\epsilon}+i} \equiv w(z^{N/2}) \quad \text{for} \quad s = \text{even} \ , \\
& -i\frac{z^{N/2}+ie^{-\epsilon}}{z^{N/2}e^{-\epsilon}-i} = w(-z^{N/2}) \quad \text{for } s = \text{odd} \ , \qquad (3.3)
\end{aligned}
$$

when $|z| \leq 1$.

Here, we introduced the regularization parameter ϵ. It is designed to obtain the convergence of summations of infinite series which we perform in various manipulations (e.g. Appendix C) and should be set zero after the computations.

We draw pictures of surfaces swept by strings in Fig. 3 and Fig. 4 for case $N = 4$. They clearly show how the strings overlap each other at the interaction time. For a finite value of ϵ, the strings overlap gradually from the end points towards the mid-point, while, for $\epsilon = 0$, the overlap occurs instantaneously. The parameter ϵ has been introduced in such a way that the strings never hit the mid-point. (*c.f.* $w_s(0) = ie^{-\epsilon}$.)

As we briefly mentioned in Section 2, the existence of the out-oscillators implies, for $z \, \epsilon D_s \bigcap D_{s+1}$, $s = 0, 1, \cdots, N-1$, relations between two adjacent strings:

$$
\begin{aligned}
\alpha_{w_s}(w_s) &= \frac{dw_{s+1}}{dw_s}\alpha_{w_{s+1}}(w_{s+1}) = -\frac{w_{s+1}}{w_s}\alpha_{w_{s+1}}(w_{s+1}) \ , \\
c^{w_s}(w_s) &= \frac{dw_s}{dw_{s+1}}c^{w_{s+1}}(w_{s+1}) = -\frac{w_s}{w_{s+1}}c^{w_{s+1}}(w_{s+1}) \ , \\
b_{w_s w_s}(w_s) &= \left(\frac{dw_{s+1}}{dw_s}\right)^2 b_{w_{s+1}w_{s+1}}(w_{s+1}) = (\frac{w_{s+1}}{w_s})^2 b_{w_{s+1}w_{s+1}}(w_{s+1}) \ . \qquad (3.4)
\end{aligned}
$$

The above equations immediately imply the overlapping conditions for $p(\tau,\sigma) = w\alpha_w(w), c(\tau,\sigma) = w^{-1}c^w$ and $b(\tau,\sigma) = w^2 b_{ww}$:

$$
\begin{aligned}
p_s(\tau = \tau_0, \sigma) &= -p_{s+1}(\tau = \tau_0, \pi - \sigma) \ , \\
c_s(\tau = \tau_0, \sigma) &= -c_{s+1}(\tau = \tau_0, \pi - \sigma) \ , \\
b_s(\tau = \tau_0, \sigma) &= +b_{s+1}(\tau = \tau_0, \pi - \sigma) \ , \\
\text{for} \ \ 0 \leq \sigma \leq \pi/2 \ \ \text{and} & \qquad \tau_0 = -\epsilon\frac{1-|z|^2}{1+|z|^2} \ . \qquad (3.5)
\end{aligned}
$$

We have to keep in mind, however, that equations like (3.4) and (3.5) which relate different strings are meaningful only in the sense of matrix elements. In the next section, we shall make more specific statements on the bra and ket vectors by which the operator equations of this kind should be understood. We also remark that the overlapping condition with the regulator is not quite a delta-function type.

For the case N =odd, the mapping, $z \mapsto w_s$ given by eq. (3.2) has a cut with the branch point $z = 0$. In this case, we have to consider a Riemann surface made of two sheets glued together at the positive real axis. For the case N =even, a single z-plane is sufficient.

So far, the map is defined in the region $|z| \leq 1$. To construct Bogoliubov coefficients unambiguously, it is necessary to extend the map in the region $|z| \geq 1$. We would like to define the map so that the smoothness across $|z| = 1$ translates into the open-string boundary conditions. We also want that the image strings never hit the mid-point as well. The map equipped with these properties is

$$\begin{aligned} w_{s,im}(z) &= -i\frac{z^{N/2} - ie^{\epsilon}}{z^{N/2}e^{\epsilon} + i} \equiv w_{im}(z^{N/2}) \quad \text{for} \quad s = \text{even} \ , \\ & \quad -i\frac{z^{N/2} + ie^{\epsilon}}{z^{N/2}e^{\epsilon} - i} = w_{im}(-z^{N/2}) \quad \text{for } s = \text{odd} \ . \end{aligned} \tag{3.6}$$

Since the difference between the maps (3.3) and (3.6) is only a sign of ϵ, we suppress the subscript im from now on unless it is necessary to distinguish these two maps.

IV. Bogoliubov Coefficients—Bosonic Part

In this section, we are going to give explicit expressions for the Bogoliubov coefficients and the inverse Bogoliubov coefficients explained in Section 2. The details will be given only for the bosonic oscillator $\alpha_z(z)$.

For $r < |z| < R$, the Pompeiu formula[20] for $\alpha_z(z)$ reads

$$\alpha_z(z) = \oint_{\{|z'|=R\}} \frac{dz'}{2\pi i} \frac{\alpha_{z'}(z')}{z' - z} - \oint_{\{|z'|=r\}} \frac{dz'}{2\pi i} \frac{\alpha_{z'}(z')}{z' - z}$$

$$+ \iint_{\{r<|z'|<R\}} \frac{dz'dz'^*}{2\pi i} \frac{1}{z'-z} \frac{d}{dz'^*} \alpha_{z'}(z') \ . \tag{4.1}$$

(It is a simple consequence of the Stokes theorem applied to the region $r < |z'| < R$ with a small circle cut out around the point z. See, Fig. 5.) If it were not for the last surface integration term, eq. (4.1) would be just a Cauchy formula. The reason why we need the Pompeiu formula, a generalization of the Cauchy formula, is that the mapping (3.3) has zeroes at $z = z_s = \exp(\frac{2\pi i s}{N} + \frac{\pi i}{N})$ in the limit $\epsilon \to +0$. This implies that $d\alpha_z(z)/dz^*$ in the last term of eq. (4.1) behaves like a delta function at $z = z_s$ when we substitute the expression

$$\begin{aligned} \alpha_z(z) &= \frac{dw_s(z)}{dz} \alpha_{w_s}(w_s) \\ &= \frac{dw_s(z)}{dz} \sum_\ell \alpha_\ell^{(s)} w_s^{-\ell-1} \ \text{ for } \ z \in \mathcal{D}_s \ . \end{aligned} \tag{4.2}$$

As we already remarked, any operator equations which relate different strings should be understood in the sense of matrix elements. We specifically consider the following set of bra and ket vectors:

$$\left\{ \left\langle V \mid \alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_K}, \ \sum n_i < \infty \right\} \ , \tag{4.3}$$

$$\text{with } \left\langle V \mid \alpha_n \right. = 0 \text{ for } \forall\, n \leq 0 \ ,$$

and

$$\left\{ \alpha_{\ell_1}^{(s_1)}, \ldots, \alpha_{\ell_L}^{(s_L)} \times_{s=0}^{N-1} \mid 0 \right\rangle_s , \ \sum \ell_i < \infty \right\} \ , \tag{4.4}$$

$$\text{with } \ \alpha_\ell^{(s)} |0>_s = 0 \text{ for } \forall\, \ell \geq 1 \ .$$

An immediate consequence of the choice of the bra vectors (4.3) is that the series

$$\alpha_z(z) = \sum_n \alpha_n z^{-n-1} \tag{4.5}$$

effectively terminates at a certain negative value of n and converges for $|z| > 1$. Similarly, the series

$$\alpha_{w_s}(w_s) = \sum_\ell \alpha_\ell^{(s)} w_s^{-\ell-1} \tag{4.6}$$

converges in this sense for $|w_s| < 1$ or $\tau_s < 0$.

The right hand side of the Pompeiu formula (4.1) is now expanded for $1 < |z| < R$ as

$$\sum_{n=0}^{\infty} \oint_{\{|z'|=R\}} \frac{dz'}{2\pi i} z'^{-n-1} z^n \alpha_{z'}(z') \; + \; \sum_{n=0}^{\infty} \oint_{\{|z'|=r\}} \frac{dz'}{2\pi i} z'^{\,n} z^{-n-1} \alpha_{z'}(z') \\ + \; \sum_{n=0}^{\infty} \sum_{s=0}^{N-1} \oint_{C_s} \frac{dz'}{2\pi i} z'^{\,n} z^{-n-1} \alpha_{z'}(z') \; . \tag{4.7}$$

Here C_s is a small contour around z_s. Recall that $d\alpha_z(z)/dz^*$ has its support in the neighborhood of z_s and that $|z_s| = 1$ as $\epsilon \to +0$. Thus the convergence of the last sum is guaranteed. Putting all together, the expansion (4.5) equals (4.7), which implies

$$\alpha_n = \oint_{\{|z'|=r\}} \frac{dz'}{2\pi i} z'^n \alpha_{z'}(z') + \sum_{s=0}^{N-1} \oint_{C_s} \frac{dz'}{2\pi i} z'^{\,n} \alpha_{z'}(z') \qquad \text{for } n \geq 0 \; , \tag{4.8}$$

and

$$\alpha_n = \oint_{\{|z'|=R\}} \frac{dz'}{2\pi i} z'^n \alpha_{z'}(z') \qquad \text{for } n \leq -1 \; . \tag{4.9}$$

Decompose the contours $|z'| = r$ and $|z| = R$ around the origin into the paths as shown in Fig. 6. Then we have

$$\alpha_n = \sum_{s=0}^{N-1} \left(\int_{a_s} \frac{dz'}{2\pi i} z'^n \alpha_{z'}(z') + \oint_{C_s} \frac{dz'}{2\pi i} z'^n \alpha_{z'}(z') \right) \qquad \text{for } n \geq 0 \; , \tag{4.10}$$

and

$$\alpha_n = \sum_{s=0}^{N-1} \int_{A_s} \frac{dz'}{2\pi i} z'^n \alpha_{z'}(z') \qquad \text{for} \qquad n \leq -1 \; . \tag{4.11}$$

Note that the union of the path a_s and the contour C_s can be deformed to A_s. We may, therefore, use either eq. (4.10) or (4.11) irrespectively of the sign of n. Now that each integration path is confined inside the s-th string region $\mathcal{D}_s$, we can rewrite $\alpha_z(z)$ in the integrand by using the transformation rule for $z \in \mathcal{D}_s$,

$$\alpha_z(z) = \frac{dw_s(z)}{dz} \alpha_{w_s}(w_s) \; , \tag{4.12}$$

where $w_s(z)$ is defined by eq. (3.3). Inserting eq. (4.12) into eq. (4.10) or (4.11) with the expansion

$$\alpha_{w_s}(w_s) = \sum_{\ell} \alpha_\ell^{(s)} (w_s)^{-\ell-1} \; , \tag{4.13}$$

we have

$$\alpha_n = \sum_{s=0}^{N-1} \sum_{\ell} A_n^{(s)\,\ell} \alpha_\ell^{(s)} \quad , \qquad \text{and} \tag{4.14}$$

$$A_n^{(s)\ell} = \int_{\substack{a_s+C_s \\ \text{or } A_s}} \frac{dz'}{2\pi i} z'^n \frac{dw_s(z')}{dz'} (w_s(z'))^{-\ell-1} \ . \tag{4.15}$$

Let us turn to the inverse Bogoliubov transformation. Apply the Pompeiu formula to $\alpha_{w_s}(w_s)$ in the w_s-plane (Fig. 7). We write

$$\begin{aligned} \alpha_{w_s}(w_s) &= \oint_{\{|w'|=R\}} \frac{dw'}{2\pi i} \frac{\alpha_{w'}(w')}{w' - w_s} - \oint_{\{|w'|=r\}} \frac{dw'}{2\pi i} \frac{\alpha_{w'}(w')}{w' - w_s} \\ &\quad + \iint_D \frac{dw' dw'^*}{2\pi i} \frac{d_{w'^*} \alpha_{w'}(w')}{w' - w_s} \ . \end{aligned} \tag{4.16}$$

The integrand of the last term on the right hand side has its support only at $|w'| = 1$ in the limit $\epsilon \to +0$, as we see when the transformation rule

$$\alpha_w(w) = \frac{dz}{dw} \alpha_z(z)$$

is substituted into eq. (4.16) with $\alpha_z(z)$ being the expansion eq. (4.5).

As already noted, the series

$$\alpha_{w_s}(w_s) = \sum_{\ell} \alpha_\ell^{(s)} (w_s)^{-\ell-1}$$

effectively terminates at some positive value of ℓ in the sense of matrix elements. Both sides of eq. (4.16) are now expanded by $w_s(z)$, with $z \in \mathcal{D}_s$

$$\begin{aligned} \sum_{\ell=-\infty}^{\infty} \alpha_\ell^{(s)} (w_s)^{-\ell-1} = \quad & \sum_{\ell=0}^{\infty} \oint_{\{|w'|=R\}} \frac{dw'}{2\pi i} w_s^{\ell} w'^{-\ell-1} \alpha_{w'}(w') \\ & + \sum_{\ell=0}^{\infty} \oint_{\{w'|=r\}} \frac{dw'}{2\pi i} w_s^{-\ell-1} w'^{\ell} \alpha_{w'}(w') \\ & - \sum_{\ell=0}^{\infty} \oint_{C_{0+\infty}} \frac{dw'}{2\pi i} w'^{-\ell-1} w_s^{\ell} \alpha_{w'}(w') \ , \end{aligned} \tag{4.17}$$

where $C_{0+\infty}$ is a contour enclosing $w_s(0)$, $w_s(\infty)$ and a cut connecting these two points. (See, Fig. 7). From eq. (4.17), we obtain

$$\alpha_\ell^{(s)} = \oint_{\{|w'|=r\}} \frac{dw'}{2\pi i} w'^{\ell} \alpha_{w'}(w') \;, \qquad \text{for } \ell \geq 0 \;, \tag{4.18}$$

and

$$\alpha_\ell^{(s)} = \oint_{\{|w'|=R\}} \frac{dw'}{2\pi i} w'^{-\ell-1} \alpha_{w'}(w') - \oint_{C_{0+\infty}} \frac{dw'}{2\pi i} w'^{-\ell-1} \alpha_{w'}(w') \;, \qquad \text{for } \ell \leq -1 \;. \tag{4.19}$$

Substituting the transformation rule $\alpha_w(w_s) = \frac{dz}{dw_s}\alpha_z(z)$ into the integrands of the right hand sides of eqs. (4.18) and (4.19), we obtain

$$\alpha_\ell^{(s)} = \sum_n \Lambda_\ell^{(s)\;n} \alpha_n \;, \tag{4.20}$$

and

$$\begin{aligned} \Lambda_\ell^{(s)\;n} &= \oint_{\{|w'|=r\}} \tfrac{dw'}{2\pi i} w'^{\ell} z'^{-n-1} \tfrac{dz'}{dw'} \\ &= \oint_{\{|w'|=R\}} \tfrac{dw'}{2\pi i} w'^{\ell} z'^{-n-1} \tfrac{dz'}{dw'} \\ &- \oint_{C_{0+\infty}} \tfrac{dw'}{2\pi i} w'^{\ell} z'^{-n-1} \tfrac{dz'}{dw'} \;. \end{aligned} \tag{4.21}$$

Here, the last equality holds for all ℓ, both negative and positive, since the contour $|w'| = r$ can be deformed to the contour $|w'| = R$ plus the contour $C_{0+\infty}$.

V. Bogoliubov Coefficients—Ghost Part

It is straightforward to extend the results of the last section to the ghost and antighost, if we note the transformation properties for $z \in \mathcal{D}_s$ given in eqs. (2.23) and (2.24). We only give the results. The Bogoliubov transformations are given for the ghost c and antighost b as

$$c_n = \sum_{s=0}^{N-1} \sum_{\ell} P_n^{(s)\ \ell} c_\ell^{(s)} , \tag{5.1}$$

$$b_n = \sum_{s=0}^{N-1} \sum_{\ell} Q_n^{(s)\ \ell} b_\ell^{(s)} . \tag{5.2}$$

The inverse Bogoliubov transformations are

$$c_\ell^{(s)} = \sum_n R_\ell^{(s)\ n} c_n , \tag{5.3}$$

$$b_\ell^{(s)} = \sum_n S_\ell^{(s)\ n} b_n . \tag{5.4}$$

The coefficients have the following integral expressions

$$P_n^{(s)\ \ell} = \int_{\substack{a_s + C_s \\ or\ A_s}} \frac{dz'}{2\pi i} z'^{n-2} \frac{dz'}{dw_s(z')} (w_s(z'))^{-\ell+1} , \tag{5.5}$$

$$Q_n^{(s)\ \ell} = \int_{\substack{a_s + C_s \\ or\ A_s}} \frac{dz'}{2\pi i} z'^{n+1} \left(\frac{dw_s(z')}{dz'} \right)^2 (w_s(z'))^{-\ell-2} , \tag{5.6}$$

$$R_\ell^{(s)\ n} = \oint_{\substack{\{|w'|=r\} \\ or\ \{|w'|=R\} - C_{0+\infty}}} \frac{dw'}{2\pi i} w'^{\ell-2} \frac{dw'}{dz'} z'^{-n+1} , \tag{5.7}$$

$$S_\ell^{(s)\ n} = \oint_{\substack{\{|w'|=r\} \\ or\ \{|w'|=R\} - C_{0+\infty}}} \frac{dw'}{2\pi i} w'^{\ell+1} \left(\frac{dz'}{dw'} \right)^2 z'^{-n-2} . \tag{5.8}$$

In Appendix B, we check, for eqs. (4.15), (4.21), and (5.5)~(5.8), some of the relations among the Bogoliubov coefficients and the inverse Bogoliubov coefficients which follow from the fundamental (anti-)commutation relations. It provides us a confidence for both the way Mandelstam map was extended away from the original region $|z| \leq 1$ and the validity of the contour in the above expressions.

VI. Vertex Continued

As outlined in Section 2, the vertex state $\langle V_N \mid$ is characterized as an out-ground state defined by the oscillators in the universal bases. Let us recall that our claim implies, in the many-string bases, the equation

$$A_n^{(r)m} - \sum_{s=0}^{N-1}\sum_{\ell=1}^{\infty} A_n^{(s)-\ell}\ell N_{\ell m}^{rs} = 0, \quad n \leq 0 ,$$
$$\text{for } m \geq 0 \text{ and } r = 0, 1, \cdots N-1 . \tag{6.1}$$

The coefficients $N_{\ell m}^{rs}$ were identified in terms of inverse Bogoliubov coefficients $\Lambda_\ell^{(s)n}$. (eq.(2.33)). Having known the integral representation of $\Lambda_\ell^{(s)n}$, we can readily construct $N_{\ell m}^{rs}$ from eq. (4.21):

$$N_{\ell m}^{rs} = \frac{1}{\ell m}\oint_{C_r}\frac{dz'}{2\pi i}\oint_{C_s}\frac{dz''}{2\pi i}(w_r(z'))^{-\ell}(w_s(z''))^{-m}\frac{1}{(z'-z'')^2} . \tag{6.2}$$

(The second term of the eq. (2.33) is in fact zero as can be seen from deforming the contour.) This is nothing but the standard integral representation of the Neumann coefficients. In Appendix A, we shall show that (6.2) is the unique solution to eq.(6.1). This amounts to an explicit operator proof of our claim for the bosonic part.

Let us turn to the ghost part of the vertex ${}^{gh}\langle V_N \mid$. According to our claim, it is an $SL_2(R)$ invariant vacuum :

$$ {}^{gh}\langle V_N | c_n = {}^{gh}\langle V_N \mid \sum_{r=0}^{N-1}\sum_{\ell} P_n^{(r)\ell} c_\ell^{(r)} = 0 , \quad n \leq -2 , \tag{6.3}$$

$$ {}^{gh}\langle V_N \mid b_n = {}^{gh}\langle V_N | \sum_{s=0}^{N-1}\sum_{m} A_n^{(s)m} b_m^{(s)} = 0 , \quad n \leq 1 . \tag{6.4}$$

As a tentative attempt, we try

$$ {}^{gh}\langle V_N | = \times_{s=0}^{N-1}\ {}_s\langle \uparrow | \exp\left(\sum_{r,s=0}^{N-1}\sum_{\substack{\ell=0\\ m=1}}^{\infty} b_\ell^{(r)}\tilde{N}_{\ell m}^{rs} c_m^{(s)}\right) , \tag{6.5}$$

where ${}_s\langle \uparrow |$ is the so-called up vacuum defined by

$$ {}_s\langle \uparrow | b_m^{(s)} = 0 , \quad m \leq -1 , \tag{6.6}$$

$$ {}_r\Big\langle \uparrow \Big| \, c_\ell^{(r)} = 0 \, , \qquad \ell \le 0 \, . \tag{6.7} $$

It is necessary that the coefficients $\tilde{N}^{rs}_{\ell m}$ satisfy

$$ P_n^{(s)m} - \sum_{r=0}^{N-1} \sum_{\ell=0}^{\infty} P_n^{(r)-\ell} \tilde{N}^{rs}_{\ell m} = 0 \ , \qquad n \le -2 \ , \ \ m \ge 0 \ , \tag{6.8} $$

$$ Q_n^{(r)\ell} + \sum_{s=0}^{N-1} \sum_{m=1}^{\infty} Q_n^{(s)-m} \tilde{N}^{rs}_{\ell m} = 0 \ , \qquad n \le 1 \ , \ \ \ell \ge 1 \ . \tag{6.9} $$

It is demonstrated in Appendix A that

$$ \tilde{N}^{rs}_{\ell m} = \oint_{C_r} \frac{dz'}{2\pi i} \oint_{C_s} \frac{dz''}{2\pi i} \frac{dz'}{dw'_r} \left(\frac{dw''_s}{dz''} \right)^2 \frac{1}{z' - z''} w_r'^{-m+1} w_s''^{-\ell-2} \ , \tag{6.10} $$

constructed from $R_\ell^{(s)n}$ and $S_\ell^{(s)n}$ in the same way as eq. (6.2) was, satisfies eqs. (6.8) and (6.9) with the matrices $P_n^{(s)m}$ and $Q_n^{(r)\ell}$ being given by eqs. (5.5) and (5.6) in Section 5 respectively.

We now see that the discussion on the ghost part of the vertex and its Neumann coefficients completely parallels the one in the bosonic case. There is, however, a notable difference between eqs.(6.8) and (6.9) we just obtained and the previous one eq. (6.1) (or eq. (2.31)): The solution to eqs. (6.8) and (6.9) is no longer guaranteed to be unique. The ghost Bogoliubov coefficients in general possess kernels, and $\tilde{N}^{sr}_{m\ell}$ is determined only up to the kernels of $P_n^{(s)m}$ or $Q_n^{(r)\ell}$. In the next section, we show how the ghost vertex is, nevertheless, uniquely determined by filling up the modes which span the kernels, simultaneously removing the ambiguity of $\tilde{N}^{sr}_{m\ell}$ mentioned above.

VII. Ghost Insertion Factors

Recall that

$$ \sum_{r=0}^{N-1} \sum_{m} \ P_n^{(r)m} \, c_m^{(r)} \ \Big| \, V \Big\rangle^{gh} \ = \ 0 \ , \ \ n \ge 2 \ , \tag{7.1} $$

$$\sum_{r=0}^{N-1}\sum_{m} Q_n^{(r)m}\, b_m^{(r)} \mid V\Big\rangle^{gh} = 0 \ , \ n \geq -1 \ , \tag{7.2}$$

Let the kernel of $P_n^{(r)m}, m = 0, \ldots$ be $\{f_m^{(r)i}\}$, $i = 1, \ldots, \dim\ker P_+$, and the kernel of $Q_n^{(r)m}, m = 1, \ldots$ be $\{g_m^{(r)j}\}$ $j = 1, \ldots, \dim\ker Q_+$: namely, they satisfy

$$\sum_{r=0}^{N-1}\sum_{m=0}^{\infty} P_n^{(r)m} f_m^{(r)i} = 0 \ , \quad n \geq 2 \ , \tag{7.3}$$

$$\sum_{r=0}^{N-1}\sum_{m=1}^{\infty} Q_n^{(r)m} g_m^{(r)j} = 0 \ , \quad n \geq -1 \ . \tag{7.4}$$

Note that the range of the m summation is bounded from below.

By inspection, we see that we can place a set of modes $\sum_{s=0}^{N-1}\sum_{m=0}^{\infty} f_m^{(s)i} b_{-m}^{(s)}$ as well as $\sum_{s=0}^{N-1}\sum_{m=1}^{\infty} g_m^{(s)j} c_{-m}^{(s)}$ freely without spoiling eqs. (7.1), (7.2). As one suspects, it is these modes that are responsible for the ghost number violation. The net charge violation is going to be dictated by the index theorem. The framework based on the Bogoliubov transformations provides a concrete realization of the Atiya-Patodi-Singer index theorem. In the following analysis, we shall investigate these systematically in the present context.

It is a general and not a novel phenomenon that net charge production associated with anomalies reflects itself into the existence of the kernels of the Bogoliubov coefficients for the fermionic degrees of freedom. This has been observed, a while ago, by Christ in his study of axial vector anomaly. We adopt his analysis to the world sheet ghost degrees of freedom. A learned reader will notice that the adaptation we make is a minor one.

Without further ado, we turn to the solution of eqs. (7.1), (7.2). Guided by the result in the last section, we write $\mid V\Big\rangle^{gh}$ as

$$\mid V\Big\rangle^{gh} = exp\left(-\sum_{r,s=0}^{N-1}\ \sum_{m=0,\ell=1}^{\infty} b_{-m}^{(s)}\ \tilde{N}_{m\ell}^{sr}\ c_{-\ell}^{(r)}\right) \mid B\Big\rangle. \tag{7.5}$$

We introduce a matrix notation in order to facilitate the algebra.

For instance,

$$\left(P_+^{(r)}\right) \equiv P_n^{(r)m}, \qquad n \geq 2,\ \ m \geq 0, \qquad r = 0, \ldots, N-1 \ , \tag{7.6}$$

$$\left(Q_-^{(r)}\right) \equiv Q_n^{(r)m}, \qquad n \geq -1,\ \ m \leq 0,\ \ r = 0, \ldots, N-1 \ .$$

Here the parentheses indicate the elements of the matrices $P_+^{(r)}$ and $Q_-^{(r)}$. Also

$$c_-) \equiv c_\ell^{(r)}, \qquad \ell \leq -1 \ , \ r = 0, \ldots, N-1 \ . \tag{7.7}$$

etc.

We choose $\mid B\big\rangle$ and $\tilde{N}$ so that the term proportional to $c_+(b_+)$ and the one proportional to $c_-(b_-)$ separately vanish in eqs. (7.1), (7.2). We obtain a set of equations to be solved:

$$\begin{aligned}(-P_+\tilde{N} + P_-)c_- \mid B\big\rangle &= 0 \ , \\ P_+c_+ \mid B\big\rangle &= 0 \ , \\ (Q_+\tilde{N} + Q_-)b_- \mid B\big\rangle &= 0 \ , \\ Q_+b_+ \mid B\big\rangle &= 0 \ .\end{aligned} \tag{7.8}$$

The summations over r and s indices are also implied.

The matrices P_+ and Q_+ are not, in general, invertible. Using the fact that $\ker P_+^\dagger$ is the orthogonal complement of ImP_+, we project eq. (7.1) onto those two subspaces. We obtain

$$\tilde{N} = P_+^{-1}\,\mathcal{P}_{ImP_+}\,P_- \ \ + \ \ \ker P_+\,X \tag{7.9}$$

as well as

$$\eta^\dagger P_-\ c_- \ \mid B\big\rangle = 0 \qquad \text{for } {}^\forall\eta \ \in \ \ker P_+^\dagger \ . \tag{7.10}$$

Here, $\mathcal{P}_{ImP_+}$ is the projection operator onto ImP_+ and X is an arbitrary vector. Similarly, from eq. (7.2), we obtain

$$\tilde{N} = Q_+^{-1}\,\mathcal{P}_{ImQ_+}\,Q_- \ \ + \ \ \ker Q_+\,Y \tag{7.11}$$

as well as

$$\zeta^{\dagger} Q_{-} b_{-} \mid B\Big\rangle = 0 \qquad \text{for } {}^{\forall}\zeta \ \in \ \ker Q_{+}^{\dagger} \ . \tag{7.12}$$

Eqs. (7.10) and (7.12) demand that a set of modes $\{\{\eta^{\dagger i} P_{-} c_{-} \mid i = 1, \ldots, \dim\ker P_{+}^{\dagger}\}\}$ and $\{\{\zeta^{\dagger j} Q_{-} b_{-} \mid j = 1, \ldots, \dim\ker Q_{+}^{\dagger}\}\}$ be filled to form $\mid B\Big\rangle$. On the other hand, the second and the fourth of eq. (7.8) tell us that these modes must lie on the kernels of P_{+} and Q_{+}:

$$P_{+}(\zeta^{\dagger j} Q_{-})^{\sim} = 0, \qquad j = 1, \ldots, \dim\ker\ Q_{+}^{\dagger}, \ \text{ and} \tag{7.13}$$

$$Q_{+}(\eta^{\dagger i} P_{-})^{\sim} = 0, \qquad i = 1, \ldots \ \dim\ker\ P_{+}^{\dagger} \ . \tag{7.14}$$

These apparently superfluous requirements are seen to be compatible once we reinforce the constraints coming from unitarity (anti-commutation) relations. Those constraints further tell us that they in fact span the kernels of P_{+} and Q_{+}. All in all, we find the final form anticipated at the beginning:

$$\mid V\Big\rangle = exp\left(-\sum_{r,s=0}^{N-1} \sum_{m=0,\ell=1}^{\infty} b_{-m}^{(s)} \ \tilde{N}_{m\ell}^{sr} \, c_{-\ell}^{(r)} \right)$$

$$\Pi_{i=1}^{\dim\ker P_{+}} f^{i} \cdot b_{-} \, \Pi_{j=1}^{\dim\ker Q_{+}} g^{j} \cdot c_{-} \times_{r=0}^{N-1} \ |\uparrow\Big\rangle_{r} \ . \tag{7.15}$$

Here, $\{\{f^{i}, \ i = 1, \ldots, \dim\ker P_{+}\}\}$ and $\{\{g^{j}, \ j = 1, \ldots, \dim\ker Q_{+}\}\}$ span the kernel of P_{+} and the kernel of Q_{+} respectively. Note that the original ambiguity of $\tilde{N}$ in eq. (7.9) is completely removed in eq. (7.15) due to the nilpotent (fermionic) nature of the modes. The ghost vertex, which is nothing but the out-vacuum in our view, is uniquely determined in terms of string oscillators despite the inherent ambiguity mentioned above.

In the remainder of this section, we will explicitly determine eq. (7.15) for the type of vertices discussed in Section 3 by finding the corresponding kernels. (Only the cases $N = 1, 2, 3$ are complete.) We also describe the net charge production associated with the Bogoliubov transformations, thereby reinforcing, in our framework, the ghost number counting due to the index theorem.

To determine the form of the kernels, we evaluate, in Appendix C, the following quantities:

$$\overset{(P)}{Z_n}\left[(i)^\ell\right] \equiv \sum_{r=0}^{N-1}\sum_{\ell} P_n^{(r)\ell}(i)^\ell \;, \quad n \geq 2 \;, \tag{7.16}$$

$$\overset{(Q)}{Z_n}\left[(-1)^r(i)^\ell\right] \equiv \sum_{r=0}^{N-1}\sum_{\ell} Q_n^{(r)\ell}(-1)^r(i)^\ell \;, \quad n \geq -1 \;. \tag{7.17}$$

Here, the summation is over all integers.(Note that $i = w(0)e^\epsilon$).

The results in Appendix C are summarized as follows: eq. (7.16) is vanishing for $^\forall n \geq 2$ only when $N = 1$. Eq. (7.17) are vanishing for $^\forall n \geq -1$ only when $N = 4, 6, 8, \ldots$. It is evident that the N dependence arises from the r summation. Using eqs. (6.8) and (6.9), eqs.(7.16) and (7.17) can be rewritten as

$$\sum_{s=0}^{N-1}\sum_{m=0}^{\infty} P_n^{(s)m}\left((i)^m + \sum_{r=0}^{N-1}\sum_{\ell=1}^{\infty}\tilde{N}_{m\ell}^{sr}(i)^{-\ell}\right) \quad \text{and} \tag{7.18}$$

$$\sum_{s=0}^{N-1}\sum_{m=1}^{\infty} Q_n^{(s)m}\left((-1)^r(i)^m - \sum_{r=0}^{N-1}\sum_{\ell=0}^{\infty}\tilde{N}_{\ell m}^{rs}(-1)^r(i)^{-\ell}\right) \tag{7.19}$$

respectively.

From these results, we conclude that $f_m^{(s)} = (i)^m + \sum_{r=0}^{N-1}\sum_{\ell=1}^{\infty}\tilde{N}_{m\ell}^{sr}(i)^{-\ell}$ is a kernel for $N = 1$ case, but not for the other cases, and that $g_m^{(s)} = (-1)^r(i)^m - \sum_{r=0}^{N-1}\sum_{\ell=0}^{\infty}\tilde{N}_{\ell m}^{rs}(-1)^r$ $(i)^{-\ell}$ is a kernel for $N = 4, 6, \ldots$, but not for the other cases.

The above analysis tells us the kernels located at $z = 0$, which is an interaction point for strings. One might suspect the existence of kernels at $z = \infty$ as well, which is an image of the interaction point through the boundary conditions. To reveal this, we must have the Laurent expansion of the out-oscillators at $z = \infty$:

$$c^z(z) = -\sum_n c_n^\infty z^{n+1} \;, \tag{7.20}$$

$$b_{zz}(z) = \sum_n b_n^\infty z^{n-2} \;. \tag{7.21}$$

The corresponding Bogoliubov coefficients are obtained if one follows the procedures in Section 4, and 5 :

$$P_n^{\infty(r)m} = -\int_{A'_r} \frac{dz}{2\pi i} z^{n-2} \left(\frac{dw_{r,im}(z)}{dz}\right)^{-1} w_{r,im}^{m+1}(z) \ , \tag{7.22}$$

$$Q_n^{\infty(r)m} = \int_{A'_r} \frac{dz}{2\pi i} z^{n+1} \left(\frac{dw_{r,im}(z)}{dz}\right)^{2} w_{r,im}^{m-2}(z) \ . \tag{7.23}$$

The paths A'_r $(r = 0, 1, \cdots, N-1)$ are shown in Fig. 8. Note that $\frac{-2\pi(r+1)}{N} \leq \arg z \leq \frac{-2\pi r}{N}$. The map $w_{r,im}(z)$ is defined in eq. (3.6).

Eqs. (7.1), (7.2), (7.3), and (7.4) are now modified : We replace $P_n^{(r)m}, Q_n^{(r)m}, f_m^{(r)i}$ and $g_m^{(r)i}$ by $P_n^{\infty(r)m}, Q_n^{\infty(r)m}, f_m^{\infty(r)i}$and $g_m^{\infty(r)i}$ respectively. (The quantities $f_m^{\infty(r)i}$ and $g_m^{\infty(r)i}$ are obtained from $f_m^{(r)i}$and $g_m^{(r)i}$ by replacing $(i)^\ell$ by $(-i)^\ell$).

In Appendix C, we evaluate

$$\overset{(P^\infty)}{Z_n} \left[(-i)^\ell\right] \equiv \sum_{r=0}^{N-1} \sum_\ell P_n^{\infty(r)\ell}(-i)^\ell \ , \qquad n \geq 2 \ , \tag{7.24}$$

$$\overset{(Q^\infty)}{Z_n} \left[(-1)^r(-i)^\ell\right] \equiv \sum_{r=0}^{N-1} \sum_\ell Q_n^{\infty(r)\ell}(-1)^r(-i)^\ell \ , \quad n \geq -1 \ . \tag{7.25}$$

The results lead to the same conclusion as the one derived from eqs. (7.18), (7.19). In other words, $f_m^{\infty(s)}$($g_m^{\infty(s)}$) is a kernel if $f_m^{(s)}$($g_m^{(s)}$) is a kernel.

One can readily check that $f_m^{(s)} = (-1)^r \delta_{m,0}$ is a kernel if and only if $N = 2$. (See Appendix C.)

The above results exhaust the kernels for the cases $N = 1, 2, 3$. One must find more kernels for the cases $N \geq 4$. The systematic study requires more analysis on the in-(or over-)completeness of the basis vectors. We plan to come back to this point in a forthcoming paper [16].

Let us now turn to the ghost number violation. As is clear from the above examples, the ghost number due to out-oscillator counting and the one due to in-oscillator counting do not match. This is precisely what the index theorem predicts. The ghost number operator $\hat{g}$ is defined to be:

$$\hat{g} \equiv -\sum_{r=0}^{N-1} \oint \frac{dw_r}{2\pi i} : b_{w_r w_r}(w_r) c^{w_r}(w_r) :$$

$$\equiv \sum_{r=0}^{N-1}\left(\sum_{\ell\geq 1}c_{-\ell}^{(r)}b_{\ell}^{(r)}-\sum_{\ell\geq 1}b_{-\ell}^{(r)}c_{\ell}^{(r)} \;+\frac{1}{2}(c_{0}^{(r)}b_{0}^{(r)}-b_{0}^{(r)}c_{0}^{(r)})\right) \tag{7.26}$$

$$= \sum_{r=0}^{N-1}\left(\sum_{\ell\geq q-1}c_{-\ell}^{(r)}b_{\ell}^{(r)}-\sum_{\ell\geq 2-q}b_{-\ell}^{(r)}c_{\ell}^{(r)}\right) \;+\; (q-3/2)\,N\;.$$

Let us suppose that the in-ground state satisfies

$$b_{\ell}^{(r)}\mid 0,\, q\Big\rangle_{r}=0,\;\; \ell\geq q-1\;, \tag{7.27}$$

$$c_{\ell}^{(r)}\mid 0,\, q\Big\rangle_{r}=0,\;\; \ell\geq 2-q\;.$$

It has ghost number $g^{(in)}=(q-3/2)N$. The present convention corresponds to $q=2$. *i.e.* $\mid 0,\, q\Big\rangle_{r}=\mid\uparrow\Big\rangle_{r}$.

Converting the expression into out-oscillators, we obtain

$$\hat{g}\;=\;D_{N}(q)\;+\;(q-3/2)N\;+\;n.o.t.\;. \tag{7.28}$$

Here, *n.o.t.* means normal ordered terms with respect to $SL(2,R)$ invariant out-ground state, and

$$D_{N}(q)\;=\;\sum_{r=0}^{N-1}\left(\sum_{n\geq 2}\sum_{\ell\geq q-1}-\sum_{n\leq 1}\sum_{\ell\leq q-2}\right)R_{-\ell}^{(r)n}\,S_{\ell}^{(r)-n}\;. \tag{7.29}$$

The actual evaluation of $D_{N}(q)$ is described here only qualitatively since rather unexpected complications have prevented us from proving the result with a complete rigor. Substituting eqs. (5.7) and (5.8) into eq. (7.29), we obtain an integral representation for $D_{N}(q)$:

$$2\oint_{i}\frac{dy}{2\pi i}\left(\frac{dw}{dy}\right)^{2}yw^{-2}\oint_{i}\frac{dy'}{2\pi i}\left(\frac{dw'}{dy'}\right)^{-1}y'^{-2}w'$$

$$\left(\sum_{\ell\geq q-1}\sum_{n\geq 2}-\sum_{\ell\leq q-2}\sum_{n\leq 1}\right)(y'/y)^{2n/N}(w'/w)^{\ell}\;. \tag{7.30}$$

Here, we have used the same change of variable used in Appendix A. One can convince oneself that this expression would be vanishing if one does not take into account the regulator: after obtaining the closed contour expression, one sees that there are two terms cancelling each other. The nonvanishing result for eq. (7.29) must be due to the careful treatment of the integrations and the convergence of the summations, which is done by our regulated conformal mapping *i.e.* eqs. (3.3) and (3.6).

In fact, the specification of the regulator tells us that we have to treat the integrand separately, depending on $|y|,\ |y'| \geq$ or $\leq\ 1$. When this is done, one sees that, in two of the above four cases, the apparent double pole one obtains after performing the summations in eq. (7.30) is really two simple poles separated by ϵ. This in turn gives rise to an asymmetry to the integrand, leading to a nonvanishing result for eq. (7.29). We plan to give a full account soon in a forthcoming paper.

The result we state with some confidence is

$$D_N(q) \;=\; -qN + 3N - 3 \ . \tag{7.31}$$

The vertex, therefore, has ghost number $g^{(out)} = 3/2N - 3$, and

$$\begin{aligned} \triangle g \equiv g^{(out)} - g^{(in)} \;&=\; (3-q)N - 3 \;=\; D_N(q) \\ &=\; N - 3 \qquad \text{for} \quad q = 2, \end{aligned} \tag{7.32}$$

which is the number of net insertions one has to make in the canonical formula eq. (7.15). Eq. (7.32) states the ghost number counting due to the index theorem.

VIII. BRST Invariance of the Vertex from the Universal Viewpoint

In this section, we are going to calculate the BRST anomaly under the Bogoliubov transformation induced by the general conformal mapping, $z \mapsto w_r(z)$.

In the many-string bases, the BRST charge Q_B is a sum of the BRST change $Q_B^{(s)}$

defined for each string :

$$Q_B = \sum_{s=0}^{N-1} Q_B^{(s)} \ . \tag{8.1}$$

The expression for $Q_B^{(s)}$ is by now standard :

$$\begin{aligned} Q_B^{(s)} &= \sum_{\ell=1}^{\infty}(c_{-\ell}^{(s)}\mathcal{L}_{\ell}^{(s)} + \mathcal{L}_{-\ell}^{(s)}c_{\ell}^{(s)}) + c_0^{(s)}(:\mathcal{L}_0^{(s)}: -1) \ , &(8.2)\\ \mathcal{L}_{\ell}^{(s)} &= L_{\ell}^{(s)X} \ + \ \frac{1}{2}L_{\ell}^{(s)gh} \ , &(8.3)\\ L_{\ell}^{(s)X} &= \oint \frac{dw}{2\pi i}w^{\ell+1}(\frac{1}{2}\alpha_w(w)^2) \ , &(8.4)\\ L_{\ell}^{(s)gh} &= \oint \frac{dw}{2\pi i}w^{\ell+1}(c^w d_w b_{ww} + 2d_w c^w b_{ww}) \ . &(8.5)\end{aligned}$$

In eq. (8.2), the normal ordering : : is taken with respect to the so-called down-vacuum $|\downarrow>$ defined by

$$\begin{aligned} c_{\ell}^{(s)} \, |\downarrow> &= 0 \qquad \text{for} \quad \ell \geq 1 \ , &(8.6)\\ b_{\ell}^{(s)} \, |\downarrow> &= 0 \qquad \text{for} \quad \ell \geq 0 \ . &(8.7)\end{aligned}$$

By using the inverse Bogoliubov transformations,

$$\begin{aligned} \alpha_{\ell}^{(s)} &= \sum_n \Lambda_{\ell}^{(s)n}\alpha_n \ , \\ c_{\ell}^{(s)} &= \sum_n R_{\ell}^{(s)n}c_n \ , \\ \text{and} \quad b_{\ell}^{(s)} &= \sum_n S_{\ell}^{(s)n}b_n \ , &(8.8)\end{aligned}$$

we write the right hand side of eq. (8.2) in terms of the oscillators in the universal bases, α_n, c_n, and b_n.

Since the operators in eq. (8.2) are not normal ordered with respect to the out-bases (the universal bases), but with respect to the in-bases (many-string bases), we have to reorganize the orderings of the operators in order to see how Q_B acts on the $SL(2,R)$-invariant vacuum given in terms of the out-bases by eq. (2.25).

First consider the case of $L_\ell^{(s)X}$. It is easy to see that

$$L_\ell^{(s)X} - :L_\ell^{(s)X}:^{out} = \frac{1}{2}\oint \frac{dw}{2\pi i} w^{\ell+1}[\alpha^{(+)}(w),\ \alpha^{(-)}(w)] \equiv \Delta_\ell^X \ , \tag{8.9}$$

where the positive and negative frequency parts $\alpha^{(+)}(w)$ and $\alpha^{(-)}(w)$ with respect to the $SL(2,R)$ state are respectively given by

$$\alpha^{(+)}(w) = \sum_\ell w^{-\ell-1} \sum_{n\geq 0} \Lambda_\ell^{(s)n} \alpha_n \ , \tag{8.10}$$

$$\alpha^{(-)}(w) = \sum_\ell w^{-\ell-1} \sum_{n\leq -1} \Lambda_\ell^{(s)n} \alpha_n \ . \tag{8.11}$$

Using the integral representation for $\Lambda_\ell^{(s)n}$ (eq. (4.21)), we obtain

$$\Delta_\ell^{(s)X} = \frac{d}{2}\sum_{\ell',\ell''} \delta_{\ell+\ell'+\ell'',0} \oint_{c_s} \frac{dz'}{2\pi i} \oint_{c_s} \frac{dz''}{2\pi i} w'^{-\ell'} w''^{-\ell''} \frac{1}{(z'-z'')^2} \ . \tag{8.12}$$

Similarly, we have, for ghosts,

$$\begin{aligned} L_\ell^{(s)gh} - :L_\ell^{(s)gh}:^{out} &= -\oint \frac{dw}{2\pi i} w^{\ell+1} (\{d_w b^{(+)}(w), c^{(-)}(w)\} + 2\{b^{(+)}(w), d_w c^{(-)}(w)\}) \\ &\equiv \Delta_\ell^{gh} \ , \end{aligned} \tag{8.13}$$

$$\begin{aligned} \Delta_\ell^{(s)gh} &= -\sum_{\ell',\ell''} \delta_{\ell+\ell'+\ell'',0} \oint_{c_s} \frac{dz'}{2\pi i} \oint_{c_s} \frac{dz''}{2\pi i} w'^{-\ell'+1} w''^{-\ell''-2} \\ &\quad \cdot \frac{dz'}{dw'} \left(\frac{dw''}{dz''}\right)^2 \frac{1}{z'-z''} (\ell' + 2\ell'') \ . \end{aligned} \tag{8.14}$$

Before proceeding further, we find it more convenient to use an alternative but an equivalent expression for $Q_B^{(s)}$:

$$Q_B^{(s)} = \sum_{\ell=2}^{\infty} c_{-\ell}^{(s)} \mathcal{L}_\ell^{(s)} + \sum_{\ell=-1}^{\infty} \mathcal{L}_{-\ell}^{(s)} c_\ell^{(s)} - \frac{1}{2} c_0^{(s)} \ . \tag{8.15}$$

Here

$$\mathcal{L}_{\pm\ell}^{(s)} =: \mathcal{L}_{\pm\ell}^{(s)}:^{out} + \Delta_{\pm\ell}^{(s)X} + \frac{1}{2}\Delta_{\pm\ell}^{(s)gh} \ , \quad \text{and} \quad c_{\mp\ell}^{(s)} = \sum_n \Lambda_{\mp\ell}^{(s)n} c_n \ .$$

We rewrite $Q_B^{(s)}$ as

$$
\begin{aligned}
Q_B^{(s)} &= \sum_{\ell=2}^{\infty}\sum_n R_{-\ell}^{(s)n} c_n (:\mathcal{L}_\ell^{(s)}: + \Delta_\ell^{(s)X} + \frac{1}{2}\Delta_\ell^{(s)gh}) \\
&\quad + \sum_{\ell=-1}^{\infty} (:\mathcal{L}_{-\ell}^{(s)}: + \Delta_{-\ell}^{(s)X} + \frac{1}{2}\Delta_{-\ell}^{(s)gh}) \sum_n R_\ell^{(s)n} c_n - \frac{1}{2}\sum_n R_0^{(s)n} c_n \,. \quad (8.16)
\end{aligned}
$$

The right hand side of eq. (8.16) is further rewritten as

$$
\begin{aligned}
&\sum_{\ell=2}^{\infty}\left(\sum_{n\le 1} R_{-\ell}^{(s)n} c_n :\mathcal{L}_\ell^{(s)}:^{out} + \sum_{n\ge 2} R_{-\ell}^{(s)n} :\mathcal{L}_{-\ell}^{(s)}:^{out} c_n\right) \\
&+ \sum_{\ell=-1}^{\infty}\left(:\mathcal{L}_{-\ell}^{(s)}:^{out} \sum_{n\ge 2} R_\ell^{(s)n} c_n + \sum_{n\le 1} R_\ell^{(s)n} c_n :\mathcal{L}_{-\ell}^{(s)}:^{out}\right) \\
&+ \sum_{\ell=2}^{\infty}\sum_{n\ge 2} R_{-\ell}^{(s)n}[c_n, \mathcal{L}_\ell^{(s)}] + \sum_{\ell=-1}^{\infty}\sum_{n\le 1} R_\ell^{(s)n}[\mathcal{L}_{-\ell}^{(s)}, c_n] \\
&+ \sum_{\ell=2}^{\infty}\sum_n R_{-\ell}^{(s)n} c_n (\Delta_\ell^{(s)X} + \frac{1}{2}\Delta_\ell^{(s)gh}) + \sum_{\ell=-1}^{\infty}\sum_n (\Delta_{-\ell}^{(s)X} + \frac{1}{2}\Delta_{-\ell}^{(s)gh}) R_\ell^{(s)n} c_n \\
&- \frac{1}{2}\sum_n R_\ell^{(s)n} c_n \,. \quad (8.17)
\end{aligned}
$$

The first two lines have normal ordered forms with respect to $\mid V\rangle$, which we write as $: Q_B^{(s)} :^{out}$. The remaining terms are what we call BRST anomaly under the Bogoliubov transformations.

We can easily check that

$$
[c_n, \mathcal{L}_\ell^{(s)}] = \frac{1}{2}[c_n, L_\ell^{(s)gh}] = -\frac{1}{2}\sum_m S_m^{(s)-n}(m-2\ell) c_{\ell-m}^{(s)} \ , \quad (8.18)
$$

and

$$
[\mathcal{L}_{-\ell}^{(s)}, c_{-n}] = \frac{1}{2}[L_{-\ell}^{(s)gh}, c_{-n}] = \frac{1}{2}\sum_m S_m^{(s)n}(m+2\ell) c_{-\ell-m}^{(s)} \quad (8.19)
$$

These should be substituted into eq. (8.17).

Therefore, the BRST anomaly is given by

$$
Q_B - :Q_B:^{out} = (I) + (II) + (III) + (IV) \ , \quad (8.20)
$$

$$(I) = \sum_n \sum_{s=0}^{N-1} \left(\sum_{\ell=2}^{\infty} R^n_{-\ell} \Delta^X_\ell + \sum_{\ell=-1}^{\infty} R^n_\ell \Delta^X_{-\ell} \right) c_n \ , \tag{8.21}$$

$$(II) = \frac{1}{2} \sum_n \sum_{s=0}^{N-1} \left(\sum_{\ell=2}^{\infty} R^n_{-\ell} \Delta^{gh}_\ell + \sum_{\ell=-1}^{\infty} R^n_\ell \Delta^{gh}_{-\ell} \right) c_n \ , \tag{8.22}$$

$$\begin{aligned}(III) = \frac{1}{2} \sum_{s=0}^{N-1} \Bigg(& \sum_{n=2}^{\infty} \sum_{\ell=2}^{\infty} \sum_m R^{(s)n}_{-\ell} S^{(s)-n}_m (2\ell - m) c^{(s)}_{\ell-m} \\ & + \sum_{n=-1}^{\infty} \sum_{\ell=-1}^{\infty} \sum_m R^{(s)-n}_\ell S^{(s)n}_{-\ell-m} (2\ell + m) c^{(s)}_{-\ell-m} \Bigg) \ ,\end{aligned} \tag{8.23}$$

$$(IV) = \sum_{s=0}^{N-1} \sum_m R^{(s)-m}_0 c^{(s)}_m \ . \tag{8.24}$$

After lengthly computations (see Appendix D for the outline), we obtain

$$(I) = \sum_{s=0}^{N-1} \oint_0 \frac{dz}{2\pi i} c^z(z) \frac{d}{2} \left\{ -\frac{1}{6} \frac{\dddot{w}_s}{\dot{w}_s} + \frac{1}{4} \frac{\ddot{w}_s^{\,2}}{\dot{w}_s^{\,2}} \right\} , \tag{8.25}$$

$$(II) = (III) = \sum_{s=0}^{N-1} \oint_0 \frac{dz}{2\pi i} c^z(z) \left\{ \frac{13}{12} \frac{\dddot{w}_s}{\dot{w}_s} - \frac{13}{8} \frac{\ddot{w}_s^{\,2}}{\dot{w}_s^{\,2}} \right\} , \tag{8.26}$$

$$(IV) = 0 \ , \tag{8.27}$$

where $\dot{w}_s = \frac{dw_s}{dz}$, etc . The final expression for the BRST anomaly is

$$Q_B - :Q_B:^{out} = -\frac{d-26}{12} \sum_{s=0}^{N-1} \oint_0 \frac{dz}{2\pi i} c^z(z) \cdot [w_s] \ , \tag{8.28}$$

with $[w_s] \equiv \frac{\dddot{w}_s}{\dot{w}_s} - \frac{3}{2}\frac{\ddot{w}_s^{\,2}}{\dot{w}_s^{\,2}}$ being the Schwarzian derivative. Equation (8.28) is a main result of this section.

As we already stated in the introduction, the BRST anomaly eq. (8.28) vanishes in $d = 26$. This immediately implies

$$\sum_{s=0}^{N-1} Q^{(s)}_B \mid V \Big\rangle =: Q_B :^{out} \mid V \Big\rangle = 0 \ . \tag{8.29}$$

This completes the proof of the BRST invariance of the vertex. We must emphasize that this is a very general proof in the sense that it is valid for any vertex unambiguously specified by the conformal map.[3] We need not repeat a proof whenever we are given a slightly different form for the vertex.

Our result eq. (8.28) has a following simple interpretation : as is well-known, the BRST charge Q_B is a generator of a gauge-fixed symmetry replacing the conformal transformations. On the other hand, our Bogoliubov transformation is induced from the conformal (Mandelstam) mapping. This implies that the quantity we calculated is essentially a conformal transformation of the BRST charge, which should behave like ${Q_B}^2$. On this ground, we argue that the BRST invariance of the vertex is in agreement with the nilpotency of the BRST charge in $d = 26$.

Obviously, the BRST anomaly vanishes in any dimension if $w_s \in SL(2, R)$, since the Schwarzian derivative vanishes in that case. It has to be so, because the $SL(2, R)$ transformation corresponds to the free propagation of a string.

A final remark is in order. The result, by no means, implies that the BRST charge is *form invariant under the Bogoliubov transformations. It is not.* This is why the $SL(2R)$ invariant out-ground state is consistent with the BRST invariance which summarizes the Virasoro gauge conditions as well as the on-shell condition for the asymptotic string configurations.

[3] As is mentioned in the introduction, the improved proof of ref. (19) should not be considered as universal. Also their treatment of the out-bases is rather different from ours: They do not consider the universal string bases as operator valued analytic first-quantized field. We do for a specific set of the matrix elements.

IX. Conclusion and Outlook

In this paper, we have put forward our idea that the string vertex is in general a Bogoliubov-transformed vacuum state unambiguously specified by the Mandelstam map. Taking Witten string field theory as an example, we rigorously proved the correctness of this idea.The results are already stated in the introduction. Here,we would like to make several additional comments briefly.

The essential spirit of our paper has been how to characterize string interactions, given two sets of asymptotic bases. A nontrivial change of the two local sets of infinite dimensional bases is given by the Bogoliubov coefficients. In this respect, it is worth mentioning that the results in Section 7 and Section 8 are geometrical in nature. This was made possible only by dismissing the delta-function overlap or connectivity conditions. It might be interesting to see how our approach is applicable to string field theory other than the one based on the BRST framework: the geometrical significance will appear in a more transparent way.

As we know, the notion of the vertex is really a first quantized concept. We suspect, therefore, that the range of application of the Bogoliubov transformation approach is considerably larger than the application to string field theory, which we have investigated here. A possible relation to ref.(21) in the operator formulation of the first quantized closed string multiloop amplitudes needs further investigation. It might provide a hint to what a closed string field theory must be. Needless to say, the usefulness of our approach for various calculations in string field theory such as off-shell amplitudes must be fully examined. It is our hope that the approach presented here sheds light on the geometrical understanding of string dynamics, by simultaneously providing a useful and unified description of string interactions.

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Note added

Related preprints we received after the completion of the paper include

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A. Leclair, M.E. Peskin and C.R. Preitschopf, SLAC-PUB 4306, 4307

APPENDIX A

Here we are going to give an explicit proof of the relation among the Bogoliubov coefficients $A_n^{(r)\ell}$ quoted in the text (eqs. (2.31) and (6.1)):

$$\sum_{r=0}^{N-1} \sum_{\ell=1}^{\infty} A_n^{(r)-\ell} \ell N_{\ell m}^{sr} = A_n^{(s)m} , \quad n \leq 0 . \tag{A.1}$$

with $N_{\ell m}^{sr}$ being the Neumann coefficients.

In order to facilitate the manipulations, we rewrite the integral expression for the Bogoliubov coefficients as

$$\begin{aligned} A_n^{(r)m} &= \int_A \frac{dz}{2\pi i} \frac{dw}{dz} z^n w^{-\ell-1} = A_n{}^\ell e^{\frac{2\pi i}{N} nr} , \\ \text{and} \quad A_n{}^\ell &= \int_A \frac{dy}{2\pi i} \frac{dw}{dy} y^{\frac{2}{N} n} w^{-\ell-1} . \end{aligned} \tag{A.2}$$

Here, we have introduced the new integration variable y defined for the r-th string by

$$z = e^{\frac{2\pi i}{N} r} y^{\frac{2}{N}} \quad , \quad 0 \leq \arg\, y \leq \pi . \tag{A.3}$$

Then we have

$$w = -i \, \frac{y-i}{y+i} \ , \quad \text{for all } r = 0, 1, \cdots, N-1 . \tag{A.4}$$

The integration path A in $(A.2)$ in the y-plane is given in Fig. 9. Note that the r dependence of $A_r^{(r)m}$ is factorized in eq. (A.2).

Similarly, the Neumann coefficients can be rewritten as follows :

$$\begin{aligned} \ell N_{m\ell}^{s\,r} &= \frac{1}{m} \oint_{z_r} \frac{dz'}{2\pi i} \oint_{z_s} \frac{dz''}{2\pi i} w_r'^{-\ell} w_s''^{-m} \frac{d}{dz''} \left(\frac{1}{z' - z''} \right) \\ &= \oint_{z_r} \frac{dz'}{2\pi i} \oint_{z_s} \frac{dz''}{2\pi i} w_r'^{-\ell} w_s''^{-m-1} \frac{1}{z' - z''} \\ &= \oint_i \frac{dy'}{2\pi i} y'^{\frac{2}{N}-1} \oint_i \frac{dy''}{2\pi i} w'^{-\ell} w''^{-m-1} \frac{dw''}{dy''} \times \\ &\quad \left[\begin{array}{l} y'^{-\frac{2}{N}} \sum_{k=0}^{\infty} \left(\frac{y''}{y'} \right)^{\frac{2}{N} k} e^{\frac{2\pi i}{N}(s-r)k} \ \Theta(|y'| > |y''|) \\ -y''^{-\frac{2}{N}} \sum_{k=0}^{\infty} \left(\frac{y'}{y''} \right)^{\frac{2}{N} k} e^{\frac{2\pi i}{N}(r-s)(k+1)} \ \Theta(|y''| > |y'|) \ . \end{array} \right] \end{aligned} \tag{A.5}$$

Here, we have expanded the kernel $\frac{1}{z'-z''}$ in the geometrical series, keeping in mind its convergence region which is indicated by the step function Θ. (It is solely for the purpose of the mnemonics and in fact an abuse of the notation.)

We are now going to calculate the sum:

$$\sum_{r=0}^{N-1} \sum_{\ell=1}^{\infty} A_n^{(r)-\ell} \ell N_{\ell m}^{s\ r} = \sum_{\ell=1}^{\infty} A_n^{-\ell} \left(\sum_{r=0}^{N-1} e^{\frac{2\pi i}{N} rn} \ell N_{\ell m}^{rs} \right) \quad . \tag{A.6}$$

The Fourier series over r is easy to compute and is given by

$$\sum_{r=0}^{N-1} e^{\frac{2\pi i}{N} rn} \ell N_{\ell m}^{r\ s} = e^{\frac{2\pi i}{N} sn} N \left(\frac{2}{N}\right) \oint_i \frac{dy'}{2\pi i} \oint_i \frac{dy''}{2\pi i} w'^{-\ell} w''^{-m-1} \frac{dw''}{dy''} \times$$
$$\times \left(\frac{y''}{y'}\right)^{2(\frac{n}{N} - [\frac{n}{N}])} \frac{y'}{y'^2 - y''^2} \quad , \tag{A.7}$$

with the square bracket being the integral part of the argument (Gaussian symbol). Then the sum (A.6) becomes

$$\sum_{\ell=1}^{\infty} \int_A \frac{dy}{2\pi i} \frac{dw}{dy} y^{\frac{2}{N}n} w^{\ell-1} e^{\frac{2\pi i}{N} sn} N \left(\frac{2}{N}\right) \oint_i \frac{dy'}{2\pi i} \oint_i \frac{dy''}{2\pi i} w'^{-\ell} w''^{-m-1} \frac{dw''}{dy''}$$
$$\times \; (\frac{y''}{y'})^{2(\frac{n}{N} - [\frac{n}{N}])} \frac{y'}{y'^2 - y''^2}$$
$$= \left(\frac{2}{N}\right) N \int_A \frac{dy}{2\pi i} \frac{dw}{dy} y^{\frac{2}{N}n} e^{\frac{2\pi i}{N} sn} \oint_1 \frac{dx'}{2\pi i} \oint_i \frac{dy''}{2\pi i} \frac{1}{w(y''x') - w} w''^{-m-1} \frac{dw''}{dy''} \times$$
$$x'^{2([\frac{n}{N}] - \frac{n}{N})} \frac{x'}{x'^2 - 1} \quad . \tag{A.8}$$

In the third line, we have changed the variable from y' to $x' = y'/y''$, which removes the apparent cut in the y'' planes, and the summation over ℓ converges in eq. (A.6). The x'-integration region is around 1 and does not cross the cut in the x'-plane. Hence, we can carry out the x' and y'' integration successively, obtaining

$$\frac{1}{2} \left(\frac{2}{N}\right) N \int_A \frac{dy}{2\pi i} \frac{dw}{dy} y^{\frac{2}{N}n} e^{\frac{2\pi i}{N} sn} w^m = A_n^{(s)m} \quad . \tag{A.9}$$

This is what we wanted to prove.

Next, we give an outline of the proof of the relations which hold for the ghost Bogoliubov coefficients $P_n^{(s)\ell}$, $Q_n^{(s)\ell}$, and the ghost Neumann coefficient $\tilde{N}_{n\ell}^{sr}$ used in the text (eqs. (6.8), (6.9)): that is

$$\sum_{s=0}^{N-1}\sum_{m=0}^{\infty} P_n^{(s)m}\tilde{N}_{m\ell}^{sr} = P_n^{(r)-\ell}\,, \tag{A.10}$$

$$\sum_{r=0}^{N-1}\sum_{\ell=1}^{\infty} \tilde{N}_{m\ell}^{sr}Q_n^{(r)\ell} = -Q_n^{(s)-m}\,. \tag{A.11}$$

It is convenient to express $P_n^{(s)\ell}$ and $Q_n^{(s)\ell}$ in the form

$$P_n^{(s)m} = \left(\frac{2}{N}\right)^2 \int_A \frac{dy}{2\pi i} y^{\frac{2}{N}n-2}\frac{dy}{dw}w^{1-m}\cdot e^{\frac{2\pi i}{N}ns}\,, \tag{A.12}$$

$$Q_n^{(r)m} = \left(\frac{N}{2}\right) \int_A \frac{dy}{2\pi i} y^{\frac{2}{N}n+1}\left(\frac{dw}{dy}\right)^2 w^{-2-\ell}\cdot e^{\frac{2\pi i}{N}nr}\,. \tag{A.13}$$

Here, we again changed the integration variable from z to y as we did for the bosonic Bogoliubov coefficients $A_\ell^{(s)m}$. Similarly to the bosonic Neumann coefficients $N_{m\ell}^{sr}$, the ghost Neumann coefficients are written as

$$\begin{aligned}\tilde{N}_{m\ell}^{sr} &= \frac{2}{N}\oint\frac{dy'}{2\pi i}\oint\frac{dy''}{2\pi i}\left(\frac{dy'}{dw'}\right)\left(\frac{dw''}{dy'}\right)^2 w'^{-m+1}w''^{-\ell-2}\frac{y''}{y'^2}\times \\ &\times\left[\begin{array}{l}\Theta(|y'|>|y''|)\sum\limits_{k=0}^{\infty}e^{\frac{2\pi i}{N}(r-s)(k-1)}\left(\frac{y''}{y'}\right)^{\frac{2}{N}(k-1)} \\ -\Theta(|y''|>|y'|)\sum\limits_{k=0}^{\infty}e^{\frac{2\pi i}{N}(s-r)(k+2)}\left(\frac{y'}{y''}\right)^{\frac{2}{N}(k+2)}\end{array}\right]\,.\end{aligned} \tag{A.14}$$

Here, we have expanded the kernel $(z'-z')^{-1}$ in eq. (6.10) in geometrical series and changed the variable from z to y.

It is easy to calculate the following series :

$$\begin{aligned}\sum_{s=0}^{N-1}e^{\frac{2\pi i}{N}sn}\tilde{N}_{m\ell}^{sr} &= 2e^{\frac{2\pi i}{N}rn}\oint_i\frac{dy'}{2\pi i}\oint_i\frac{dy''}{2\pi i}\frac{dy'}{dw'}\left(\frac{dw''}{dy''}\right)^2 w'^{-m+1}w''^{-\ell-2}\frac{1}{y''}\times \\ &\times\frac{1}{\left(\frac{y'}{y''}\right)^2-1}\left(\frac{y''}{y'}\right)^{2\left(\frac{n}{N}-\left[\frac{n+1}{N}\right]\right)}\,,\end{aligned} \tag{A.15}$$

$$\sum_{r=0}^{N-1}\tilde{N}_{m\ell}^{sr}e^{\frac{2\pi i}{N}rn} = 2e^{\frac{2\pi i}{N}sn}\oint_i\frac{dy'}{2\pi i}\oint_i\frac{dy''}{2\pi i}\frac{dy'}{dw'}\left(\frac{dw''}{dy''}\right)^2 w'^{-m+1}w''^{-\ell-2}\frac{y''}{y'^2}\times$$

$$\times \frac{1}{1-\left(\frac{y''}{y'}\right)^2} \left(\frac{y''}{y'}\right)^{2\left(\left[\frac{n-2}{N}\right]-\frac{n}{N}+1\right)} \quad . \tag{A.16}$$

We go on to calculate the left hand side of eq. (A.10) :

$$\begin{aligned}
&\sum_{s=0}^{N-1} \sum_{m=0}^{\infty} P_n^{(s)m} \tilde{N}_{m\ell}^{sr} \\
&= \sum_{m=0}^{\infty} \left(\frac{2}{N}\right)^2 \int_C \frac{dy}{2\pi i} y^{\frac{2}{N}n-2} \frac{dy}{dw} w^{1-m} \cdot 2e^{\frac{2\pi i}{N}rn} \oint_i \frac{dx'}{2\pi i} \oint_i \frac{dy''}{2\pi i} \frac{dx'}{dw'} \left(\frac{dw''}{dy''}\right)^2 \times \\
&\quad \times w'^{-m+1} w''^{-\ell-2} \frac{y''}{x'^2-1} x'^{2\left(\left[\frac{n+1}{N}\right]-\frac{n}{N}\right)} , \\
&= \left(\frac{2}{N}\right)^2 \int_C \frac{dy}{2\pi i} y^{\frac{2}{N}n-2} \frac{dy}{dw} e^{\frac{2\pi i}{N}rn} \oint_1 \frac{dx'}{2\pi i} \oint_{\frac{-i}{x'}} \frac{dy''}{2\pi i} \frac{ww'}{1-\frac{1}{ww'}} \left(\frac{dw''}{dy''}\right)^2 \quad .
\end{aligned} \tag{A.17}$$

Here, in the second line, we have changed the integration variable, setting $y' = x'y''$, which enables us to remove the apparent cut in the y''-plane. We have also deformed the contour for the y''-integration to the one around $-i/x'$, so that $|w'|$ becomes large and therefore the summation over index m converges. Then we can easily carry out the y''-integration by picking up the pole at $w' = w^{-1}$. Noting that the contour for the integration over x' is around 1, which is away from the cut along the negative real axis. The result is

$$\left(\frac{2}{N}\right)^2 \int_A \frac{dy}{2\pi i} y^{\frac{2}{N}n-2} \frac{dy}{dw} w^{\ell+1} \cdot e^{\frac{2\pi i}{N}m} = P_n^{(r)-\ell} \quad ,$$

which is what we wanted to show.

The manipulation for the proof of (A.9) is completely parallel to the one explained in (A.15) with the use of (A.16).

The uniqueness of the solution for $A_n^{(s)\ell}$ of eq. (A.1) comes from the fact that the matrix ($A_n^{(s)\ell}$, $n, \ell \leq 0$) has no kernel and therefore is invertible. This can be seen, for example, from the completeness relation shown in Appendix B,

$$\sum_{\ell} \sum_{s=0}^{N-1} A_n^{(s)\ell} \ell A_m^{(s)-\ell} = n\delta_{n+m,0} \quad , \tag{A.18}$$

or with

$$a_n^{(s)\ell} \equiv \frac{\sqrt{|\ell|}}{\sqrt{|n|}} A_n^{(s)-\ell}, \quad b_n^{(s)\ell} \equiv \frac{\sqrt{|\ell|}}{\sqrt{|n|}} A_n^{(s)\ell} \ , \tag{A.19}$$

$$\sum_{s=0}^{N-1} \left(a^{(s)} a^{(s)\dagger} - b^{(s)} b^{(s)\dagger} \right) = 1 \ , \tag{A.20}$$

in the matrix notation. It is now clear that $\sum_{s=0}^{N-1} a^{(s)} a^{(s)\dagger}$ is positive definite. Therefore, there is no kernel for $a^{(s)}$ and for the Bogoliubov coefficients $A_n^{(s)\ell}\ n, \ \ \ell \leq 0$.

APPENDIX B

We are going to present an explicit proof of the completeness relations:

$$\sum_{\ell} \sum_{s=0}^{N-1} A_n^{(s)\ell} \ell A_m^{(s)-\ell} \ = \ n \delta_{n+m,\,0} \ , \tag{B.1}$$

$$\sum_{\ell} \sum_{s=0}^{N-1} P_n^{(s)\ell} Q_m^{(s)-\ell} \ = \ \delta_{n+m,\,0} \ . \tag{B.2}$$

for bosonic and ghost oscillators, repsectively. The equations $(B.1)$ and $(B.2)$ are also consistency conditions which follow from the commutation relations in the universal and many-string bases : under eq. (4.14), the transformations

$$\begin{aligned} [\alpha_n, \alpha_m] \ &= \ n \ \delta_{n+m,\,0} \\ \text{and} \quad \left[\alpha_\ell^{(r)}, \ \alpha_m^{(s)}\right] \ &= \ \delta_{r,s} \ \ell \ \delta_{\ell+m,\,0} \end{aligned} \tag{B.3}$$

imply (B.1). Similarly, the ghost anticommutation relations

$$\begin{aligned} \{c_n, \ b_m\} \ &= \ \delta_{n+m,\,0} \ , \\ \text{and} \quad \{c_\ell^{(r)}, \ b_m^{(s)}\} \ &= \ \delta_{r,s} \delta_{\ell+m,\,0} \ , \end{aligned} \tag{B.4}$$

imply (B.2) because they are related by eqs. (5.1)and (5.2).

The check of (B.1) and (B.2) partially supports the correctness of our choice of the integration paths given in Section 4.

Let us start with the integral expression:

$$A_n^{(s)\ell} = \int_{A'} \frac{dy'}{2\pi i}\frac{dw'}{dy'} y'^{\frac{2}{N}n} w'^{-\ell-1} \cdot e^{\frac{2\pi i}{N}ns} \quad , \tag{B.5}$$

with the path A' running from 1 to -1 via upper half plane. Performing the summation over s, we find

$$\begin{aligned} &\sum_\ell \sum_{s=0}^{N-1} A_n^{(s)\ell} \ell A_m^{(s)-\ell} \\ &= N \sum_\ell \ell \int_{A'} \frac{dy'}{2\pi i}\frac{dw'}{dy'} y'^{\frac{2}{N}n} w'^{-\ell-1} \int_{A''} \frac{dy''}{2\pi i}\frac{dw''}{dy'} y''^{\frac{2}{N}m} w''^{\ell-1} \quad , \end{aligned} \tag{B.6}$$

if $n+m$ is a multiple of N. Otherwise it vanishes. Let $n+m$ be a multiple of N, and n be positive and m be negative without loss of generality. (If n and m have same sign, one can easily show that (B.6) vanishes.)

The right hand side of $(B.6)$ becomes

$$N \int_{A'} \frac{dy'}{2\pi i}\frac{dw'}{dy'} y'^{\frac{2}{N}n} w'^{-\ell-1} \int_{A''} \frac{dy''}{2\pi i}\frac{d}{dy''} \left(\sum_{\ell=0}^{\infty} w'^{-\ell-1} w''^{\ell} + \sum_{\ell=-\infty}^{-1} w'^{-\ell-1} w''^{\ell} \right) \quad . \tag{B.7}$$

We deform the paths so that the geometric series converge. The result is

$$N \int_{A''} \frac{dy''}{2\pi i} y''^{\frac{2}{N}m} \frac{d}{dy''} \left(\int_{A_>} - \int_{A_<} \right) \frac{dw'}{2\pi i} y'^{\frac{2}{N}n} \frac{1}{w'-w''} \quad . \tag{B.8}$$

Here the paths are shown in Fig. 10. Evidently, the path $A_> - A_<$ develop into a closed contour enclosing w''. We obtain

$$2n \int_{A''} \frac{dy''}{2\pi i} y''^{\frac{2}{N}(n+m)-1} = 2n \int_{A''} \frac{dy''}{2\pi i} y''^{2k-1} = n\ \delta_{k,0} \quad . \tag{B.9}$$

with k being an integer. Hence we arrive at (B.1).

Now we turn to eq. (B.2). Recall eqs. (A.12) and (A.13) for the ghost Bogoliubov coefficients. We only consider the case that $n+m$ is a multiple of N since, otherwise, the left hand side vanishes owing to the Z_N Fourier summation.

We proceed by splitting the sum $\sum_\ell = \sum_{\ell\geq 2} + \sum_{\ell \leq 1}$:

$$\sum_\ell \sum_{s=0}^{N-1} P_n^{(s)\ell} Q_m^{(s)-\ell}$$

$$= 2\int_{A''}\frac{dy''}{2\pi i}\left(\int_{A_>}-\int_{A_<}\right)\frac{dw'}{2\pi i}y'^{\frac{2n}{N}-2}y''^{\frac{2m}{N}+1}\left(\frac{dw'}{dy'}\right)^{-2}\left(\frac{dw''}{dy''}\right)^{2}\frac{1}{w'-w''}$$

$$= 2\int_{A''}\frac{dy''}{2\pi i}y''^{2k-1} = \delta_{k,\,0} \ . \tag{B.10}$$

In this way, we arrive at eq. (B.2).

APPENDIX C

In this appendix, we evaluate the quantities defined in eqs. (7.16), (7.17), (7.24) and (7.25). Using the change of variables seen in Appendix A, we see that eq. (7.16) is nonvanishing, after the r summation, only when n is an integral multiple of N, *i.e.* , $n = Nn'$. Here, $n' \geq 1$ for $N \geq 2$ and $n' \geq 2$ for $N = 1$. We obtain, for these cases,

$$\overset{(P)}{Z_n}[(i)^\ell] = N(2/N)^2\int_A\frac{dy}{2\pi i}\left(\frac{dw}{dy}\right)^{-1}y^{2n'-2}\sum_\ell w^{-\ell+1}(y)(i)^\ell \ . \tag{C.1}$$

We decompose the infinite sum into two parts *i.e.* $\ell \geq 2$ and $\ell \leq 1$. For $\ell \geq 2$, we make the radius of the path very large in the w-plane to guarantee the convergence of the series. (This is possible only when one introduces the regulator ϵ.) For $\ell \leq 1$, one can deform the path across the origin. (no singularity at $y = ie^\epsilon$.) One can find a path right below the origin which guarantees the convergence of the series. (See Fig. 11). The summation of these two contributions develops into an expression with a closed contour around the origin:

$$-\frac{4}{N}\oint_0\frac{dy}{2\pi i}\left(\frac{dw}{dy}\right)^{-1}y^{2n'-2}\frac{i}{w-i} \ . \tag{C.2}$$

We set $\epsilon = 0$ at this point. It is now obvious that this quantity is nonvanishing only when $n' = 1$, which is possible for $N = 1$ only.

Let us now turn to the evaluation of eq. (7.17). The r summation again tells us that this quantity is nonvanishing if N is odd, and is vanishing for $N =$ even unless $n = (N/2)(2n'+1)$, with $n' \in \mathbf{Z}$. We, therefore, restrict ourselves to these cases.

We obtain

$$\overset{(Q)}{Z_n}\,[(-1)^r(i)^\ell] = N(N/2)\int_A \frac{dy}{2\pi i}\left(\frac{dw}{dy}\right)^2 y^{2n'+2}\sum_\ell w^{-\ell-2}(y)(i)^\ell \ . \tag{C.3}$$

Here, $n' \geq -1$ for $N = 2$ case, and $n' \geq 0$ for $N = 4, 6, 8, \cdots$. Following the same procedure for the deformation of the contour as above, we obtain

$$-\frac{N^2}{2}\oint_0 \frac{dy}{2\pi i}\left(\frac{dw}{dy}\right)^2 y^{2n'+2}\frac{i}{w-i} \ . \tag{C.4}$$

We conclude that eq. (7.17) is vanishing for $\forall n \geq -1$, only for the $N = 4, 6, 8, \cdots$ cases.

One can, in a similar way, analyse the quantities (7.24) and (7.25). For eq. (7.24), we obtain a result completely analogous to eq. (C.1) for the same cases :

$$\overset{(P^\infty)}{Z_n}\,[(-i)^\ell] = -N(2/N)^2\int_{A'} \frac{dy}{2\pi i}\left(\frac{dw_{im}}{dy}\right)^{-1} y^{2n'-2}\sum_\ell w_{im}^{\ell+1}(y)(-i)^\ell \ . \tag{C.5}$$

The path A' is shown in Fig. 12. Here, we made a change of variable $z = e^{-2\pi i r/N}y^{2/N}$ $-\pi \leq y \leq 0$. Also, $w_{im} = -i\frac{y-ie^\epsilon}{ye^\epsilon+i}$.

Again, we decompose the infinite sum into two parts : $\ell \geq -1$ and $\ell \leq -2$. For $\ell \geq -1$, we let the radius of the path be very large in the lower half plane to make the series convergent. For $\ell \leq -2$, we can deform the contour across $y = -ie^{-\epsilon}$ and the origin. One finds a path right above the real axis which guarantees the convergence. We obtain

$$\frac{4}{N}\oint_0 \frac{dy}{2\pi i}\left(\frac{dw_{im}}{dy}\right)^{-1} y^{2n'-2}\frac{(-i)^{-2}}{w_{im}(y)-i} \ , \tag{C.6}$$

leading to the same conclusion as the one derived from eq. (C.2).

Let us now turn to the quantity (7.25). After some manipulations, we obtain a result completely analogous to eq. (C.3) for the same cases.

$$\overset{(Q)}{Z_n}\,[(-1)^r(-i)^\ell] = N(N/2)\int_{A'} \frac{dy}{2\pi i}\left(\frac{dw_{im}}{dy}\right)^2 y^{2n'+2}\sum_\ell w_{im}^{\ell-2}(y)(-i)^\ell \ . \tag{C.7}$$

Following the same argument for the deformation of the contour as the one leading to eq. (C.6), we find

$$-\frac{N^2}{2}\oint_0 \frac{dy}{2\pi i}\left(\frac{d(1/w_{im})}{dy}\right)^2 y^{2n'+2}\frac{i}{w_{im}-i} \quad , \tag{C.8}$$

establishing the same conclusions the one stated on eq. (C.4).

Finally, let us calculate the following quantity:

$$\sum_{r=0}^{N-1}\sum_{r=0} P_n^{(r)\ m}(-1)^r\delta_{m,0} \quad , n \geq 2 \ . \tag{C.9}$$

Like eqs. (7.17) and (7.25), the only case of interest is N =even and $n = (N/2)(2n'+1)$. Also, $n' \geq 1$ for $N = 2$ and $n' \geq 0$ for $N = 4, 6, 8, \cdots$. For these cases, we obtain a expression with a closed contour :

$$2/N \oint \frac{dy}{2\pi i}\left(\frac{dw}{dy}\right)^{-1} y^{2n'-1}w(y) \quad , \tag{C.10}$$

which is nonvanishing only when $n' = 0$.

We finish this appendix, saying that the manipulations given here largely owe their success to the way Mandelstam map is regulated.

APPENDIX D

In this appendix, we will give an outline of the calculations of (I) - (III) in eqs. (8.21), (8.22) and (8.23), which are vital to obtaining the expression for the BRST anomaly (8.28).

First, look at

$$(I) = \sum_n \sum_{s=0}^{N-1}\left(\sum_{\ell=2}^{\infty} R_{-\ell}^{(s)n}\Delta_{\ell}^{(s)X} + \sum_{\ell=-1}^{\infty} R_{\ell}^{(s)n}\Delta_{-\ell}^{(s)X}\right) c_n \ , \tag{D.1}$$

where

$$R_{\ell}^{(s)n} = \oint_{C_s}\frac{dz}{2\pi i} w_s^{\ell-2}\left(\frac{dw_s}{dz}\right)^2 z^{1-n} \quad \text{(see, eq. (5.7) in Section 5)} \ , \tag{D.2}$$

and

$$\Delta_\ell^{(s)X} = \frac{d}{2}\sum_{\ell',\ell''} \delta_{\ell-\ell'+\ell'',0} \oint_{C_s} \frac{dz'}{2\pi i} \oint_{C_s} \frac{dz''}{2\pi i} w_s'^{-\ell'} w_s''^{-\ell''} \frac{1}{(z'-z'')^2} \ , \tag{D.3}$$

as explained in Section. 8. The quantity inside the bracket in eq. (8.1) becomes

$$\frac{d}{2}\oint_{C_s} \frac{dz}{2\pi i}\left(\frac{dw}{dz}\right)^2 z^{1-n} \left[\sum_{\ell=2}^{\infty} w^{-\ell-2} \sum_{\ell'} \oint_{C_s} \frac{dz'}{2\pi i} \oint_{C_s} \frac{dz''}{2\pi i} w'^{-\ell'} w''^{\ell+\ell'} \frac{1}{(z'-z'')^2} \right.$$
$$\left. + \sum_{\ell=-1}^{\infty} w^{\ell-2} \sum_{\ell'} \oint_{C_s} \frac{dz'}{2\pi i} \oint_{C_s} \frac{dz''}{2\pi i} w'^{-\ell'} w''^{-\ell+\ell'} \frac{1}{(z'-z'')^2} \right] \ . \tag{D.4}$$

Here, we have omitted the suffix s of w for typographical simplicity. The sum over ℓ is carried out by the suitable choice of the contours. We obtain

$$\frac{d}{2}\oint_{C_s} \frac{dz}{2\pi i}\left(\frac{dw}{dz}\right)^2 z^{1-n} \sum_{\ell'} \left\{ \oint \frac{dz'}{2\pi i} \oint \frac{dz''}{2\pi i} \frac{w^{-1} w''^2}{w - w''} \frac{1}{(z'-z'')^2} \Theta(|w| > |w''|) \right.$$
$$\left. - \oint \frac{dz'}{2\pi i} \oint \frac{dz''}{2\pi i} \frac{w^{-1} w''^2}{w - w''} w'^{-\ell'} w''^{\ell'} \frac{1}{(z'-z'')^2} \Theta(|w| < |w''|) \right\} \ .$$

The standard manipulation of contour integrals in the conformal field theory enables us to carry out the z'' integration and gives

$$\frac{d}{2}\oint_{C_s} \frac{dz}{2\pi i}\left(\frac{dw}{dz}\right)^2 z^{1-n} \sum_{\ell'} \oint \frac{dz'}{2\pi i} w^{\ell'+1} w'^{-\ell'} \left(\frac{dw}{dz}\right)^{-1} \frac{1}{(z'-z)^2} \ . \tag{D.5}$$

The same technique can also apply for the summation over ℓ' and the subsequent integration over z'. We obtain

$$\sum_{\ell=2}^{\infty} R_{-\ell}^{(s)n} \Delta_\ell^{(s)X} + \sum_{\ell=-1}^{\infty} R_\ell^{(s)n} \Delta_{-\ell}^{(s)X}$$
$$= \frac{d}{2}\oint_{C_s} \frac{dz}{2\pi i} z^{1-n} \oint_{C_z} \frac{dz'}{2\pi i} \frac{w w'}{w' - w} \frac{1}{(z'-z)^2} \frac{dw}{dz} \ . \tag{D.6}$$

with C_z being the contour around z. An elementary calculation shows that the z'-integral becomes

$$\oint_{C_z} \frac{dz'}{2\pi i} \frac{w w'}{w' - w} \frac{1}{(z'-z)^2} \frac{dw}{dz} = -\frac{1}{6}[w] \ , \tag{D.7}$$

with $[w]$ being the Schwarzian derivative of w defined by

$$[w] = \frac{d^3w}{dz^3}\left(\frac{dw}{dz}\right)^{-1} - \frac{3}{2}\left(\frac{d^2w}{dz^2}\right)^2\left(\frac{dw}{dz}\right)^{-2} \quad . \tag{D.8}$$

Hence, the right hand side of (D.6) becomes

$$-\frac{d}{12}\oint_{C_s}\frac{dz}{2\pi i}z^{1-n}\cdot[w_s] \, . \tag{D.9}$$

Going back to (D.1), we obtain

$$(I) = \sum_{s=0}^{N-1}\oint_{c_s}\frac{dz}{2\pi i}c^z(z)\left(-\frac{d}{12}[w_s]\right) \quad , \tag{D.10}$$

which is just the first line of eq. (8.23).

Let us turn to (II) in eq. (8.22),

$$(II) = \sum_n\sum_{s=0}^{N-1}\left(\sum_{\ell=2}^{\infty}R_{-\ell}^{(s)n}\Delta_\ell^{(s)gh} + \sum_{\ell=-1}^{\infty}R_\ell^{(s)n}\Delta_{-\ell}^{(s)gh}\right)c_n \quad , \tag{D.11}$$

where $R_\ell^{(s)n}$ is given in (D.2) and

$$\Delta_\ell^{(s)gh} = -\sum_{\ell',\ell''}\delta_{n+\ell'+\ell'',0}\oint_{C_s}\frac{dz'}{2\pi i}\oint_{C_s}\frac{dz''}{2\pi i}w'^{-\ell'+1}w''^{-\ell''-2}\left(\frac{dw'}{dz'}\right)^{-1}\times$$
$$\times\left(\frac{dw''}{dz''}\right)^2\frac{1}{z'-z''}(\ell'+2\ell'') \quad . \tag{D.12}$$

We can proceed to do the summation over ℓ and the integration over z'' in a parallel way to the previous case (I), and then perform the summation over ℓ'. We obtain

$$\sum_{\ell=2}^{\infty}R_{-\ell}^{(s)n}\Delta_\ell^{(s)gh} + \sum_{\ell=-1}^{\infty}R_\ell^{(s)n}\Delta_{-\ell}^{(s)gh}$$
$$\simeq \oint_{C_s}\frac{dz}{2\pi i}z^{1-n}\oint_{C_s}\frac{dz'}{2\pi i}\left(\frac{dw}{dz}\right)^2\left(\frac{dw'}{dz'}\right)^{-1}w^{-2}w'\left[\frac{-w'}{(w'-w)^2}\frac{1}{z'-z}\frac{dw}{dz}\right.$$
$$\left.-2\frac{w'}{(w'-w)(z'-z)}\frac{d^2w/dz^2}{dw/dz}-2\frac{w'}{(z'-z)^2(w'-w)}+\frac{2w'w^{-1}}{(w'-w)}\frac{dw}{dz}\frac{1}{z'-z}\right] \, . \tag{D.13}$$

The z' integration in (D.13) turns out to be

$$\oint_{C_s} \frac{dz'}{2\pi i} \cdots = \frac{13}{12}[w] - \frac{9}{2}\frac{\frac{d^2w}{dz^2}}{w} - 3 \ . \tag{D.14}$$

Hence, going back to eq. (D.11), we obtain

$$(II) = \sum_{s=0}^{N-1} \oint_{C_s} \frac{dz}{2\pi i} c^z(z) \frac{13}{12} \cdot [w_s] \ . \tag{D.15}$$

Note that the last two terms in eq. (D.14) do not contribute to (II).

Finally, we are going to calculate

$$\begin{aligned} (III) \ = \ & \frac{1}{2}\sum_{s=0}^{N-1}\left(\sum_{n=2}^{\infty}\sum_{\ell=2}^{\infty}\sum_{m} R_{-\ell}^{(s)n} S_m^{(s)-n}(2\ell - m)c_{\ell-m}^{(s)}\right. \\ & \left. + \sum_{n=-1}^{\infty}\sum_{\ell=-1}^{\infty}\sum_{m} R_{\ell}^{(s)-n} S_m^{(s)n}(2\ell + m)c_{-\ell-m}^{(s)}\right) \ . \end{aligned} \tag{D.16}$$

It is more convenient to rewrite (D.16) with an integral representation.

$$(III) \ = \ \frac{1}{2}\sum_{s=0}^{N-1} \oint_{w=0} \frac{dw}{2\pi i} J^{(s)} c^w(w) \ , \tag{D.17}$$

$$\begin{aligned} J^{(s)} \ = \ & \sum_{n=2}^{\infty}\sum_{\ell=1}^{\infty}\sum_{m} R_{-\ell}^{(s)n} S_m^{(s)-n}(2\ell - m) w^{\ell-m-2} \\ & + \sum_{n=-1}^{\infty}\sum_{\ell=-1}^{\infty}\sum_{m} R_{\ell}^{(s)-n} S_m^{(s)n}(2\ell + m) w^{-\ell-m-2} \ . \end{aligned} \tag{D.18}$$

Here, $R_\ell^{(s)n}$ is given in (D.2) and $S_m^{(s)n}$ is given by

$$S_m^{(s)n} = \oint_{C_s} \frac{dz}{2\pi i} w^{m+1} \left(\frac{dw}{dz}\right)^{-1} z^{-2-n} \ . \tag{D.19}$$

(See eq.(5.8) in Section 5). Then the sum $J^{(s)}$ becomes

$$\begin{aligned} & \sum_{n=2}^{\infty}\sum_{\ell=2}^{\infty}\sum_{m} \oint_{C_s} \frac{dz'}{2\pi i} w'^{-\ell-2} \left(\frac{dw'}{dz'}\right)^2 z'^{1-n} \oint_{C_s} \frac{dz''}{2\pi i} w''^{m+1} \times \\ & \times \left(\frac{dw''}{dz''}\right)^{-1} z''^{-2+n} w^{\ell-m-2}(2\ell - m) \end{aligned}$$

$$+\sum_{n=-1}^{\infty}\sum_{\ell=-1}^{\infty}\sum_{m}\oint_{C_s}\frac{dz'}{2\pi i}w'^{\ell-2}\left(\frac{dw'}{dz'}\right)^{2}z'^{1+n}\oint_{C_s}\frac{dz''}{2\pi i}w''^{m+1}\times$$
$$\times\left(\frac{dw''}{dz''}\right)^{-2}z''^{-2-n}w^{-\ell-m-2}(2\ell+m)\,. \tag{D.20}$$

Carrying out the summation over n and ℓ, we can proceed

$$\begin{aligned}
&\sum_{\ell=2}^{\infty}\sum_{m}\oint_{C_S}\frac{dz'}{2\pi i}w'^{-\ell-2}\left(\frac{dw'}{dz'}\right)^{2}\oint_{C_s}\frac{dz''}{2\pi i}w''^{m+1}\left(\frac{dw''}{dz''}\right)^{-1}\\
&\qquad w^{\ell-m-2}\frac{(2\ell-m)}{z'-z''}\Theta(|z'|>|z''|)\\
&\qquad+\sum_{\ell=-1}^{\infty}\sum_{m}\oint_{C_s}\frac{dz'}{2\pi i}w^{-\ell-2}\left(\frac{dw'}{dz'}\right)^{2}\oint_{C_s}\frac{dz''}{2\pi i}w''^{m+1}\left(\frac{dw''}{dz''}\right)^{-1}\\
&\qquad w^{-\ell-m-2}\frac{(2\ell+m)}{z''-z'}\Theta(|z''|>|z'|)\\
&=\sum_{m}\oint_{C_s}\frac{dz'}{2\pi i}\oint_{C_s}\frac{dz''}{2\pi i}\left(\frac{dw'}{dz'}\right)^{2}\left(\frac{dw''}{dz''}\right)^{-1}w''^{m+1}w^{-m-1}\frac{w'^{-2}}{z'-z''}\times\\
&\qquad\times\left[2\left(\frac{2ww'^{-1}}{w'-w}+\frac{w^{2}w'^{-1}}{(w-w')^{2}}\right)-\ell\frac{ww'^{-1}}{w'-w}\right][\Theta(|w'|>|w|)-\Theta(|w|>|w'|)]\,.
\end{aligned} \tag{D.21}$$

The standard deformation of contour for the z'-integration gives

$$\begin{aligned}
J^{(s)}\;=\;&-\oint_{C_s}\frac{dz''}{2\pi i}\left(\frac{dw''}{dz''}\right)^{-1}\frac{1}{z''-z}\left[-2\frac{\frac{dw}{dz}w''w^{-2}}{w''-w}\right.\\
&\left.+\frac{dw}{dz}w''w^{-1}\frac{1}{(w''-w)^{2}}+\frac{2\frac{d^2w}{dz^2}w''}{\frac{dw}{dz}w(w''-w)}+\frac{2w''w^{-1}}{(z''-z)^{2}(w''-w)}\right]\,.
\end{aligned} \tag{D.22}$$

An elementary calculation shows that

$$J^{s}=\frac{13}{6}[w]\cdot\left(\frac{dw}{dz}\right)^{-2}+\frac{2}{w^{2}}-\frac{2}{3}\frac{\frac{d^2w}{dz2}}{\left(\frac{dw}{dz}\right)^{2}w}\,. \tag{D.23}$$

Since the last two terms do not contribute to (III), we obtain

$$(III) = \sum_{s=0}^{N-1} \oint_{C_s} \frac{dz}{2\pi i} c^z(z) \frac{13}{12} [w_s] \ . \tag{D.24}$$

It is easy to show that $(IV) = 0$ (eq. (8.27)). This completes the calculations listed in eqs. (8.25), (8.26) and (8.27).

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Figure Captions

Fig. 1: Three-string overlap in the string field theory of Witten

Fig. 2: An alternative pictorial view of incoming and outgoing string states

Fig. 3: Four-string vertex in the universal z-plane. The s-th string sweeps the domain $\mathcal{D}_s$.

Fig. 4: The domain $\mathcal{D}_s$ mapped into the $\log w_s$ plane by the Mandelstam mapping: an incoming string at $\tau_s = -\infty$ sweeps a semi-infinite strip toward the interaction point. A semi-infinite strip in the lower half plane is the image of the upper half plane.

Fig. 5: The domain in the z'-plane to which we apply the Pompeiu formula

Fig. 6: Large and small contours around the origin divided into paths to obtain the Bogoliubov coefficients

Fig. 7: The domain in the w'-plane to which we apply the Pompeiu formula: the region enclosing the cut from $w'(0)$ to $w'(\infty)$ is cut out as well as the small disk around w_s and the origin.

Fig. 8: The paths A'_s for eqs. (7.22) and (7.23) $(N = 4)$

Fig. 9: The integration path A in the y-plane

Fig. 10: The integration paths $A_>$and $A_<$ which guarantee the convergence of the summations in eq. (B.6)

Fig. 11: The integration paths which guarantee the convergence of the summations in eq. (C.1)

Fig. 12: The integration path A' in the y-plane and its deformation in eq. (C.5)

Fig. 1.

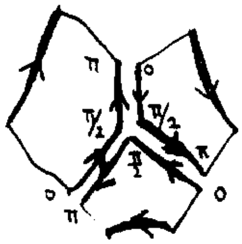


Fig. 2

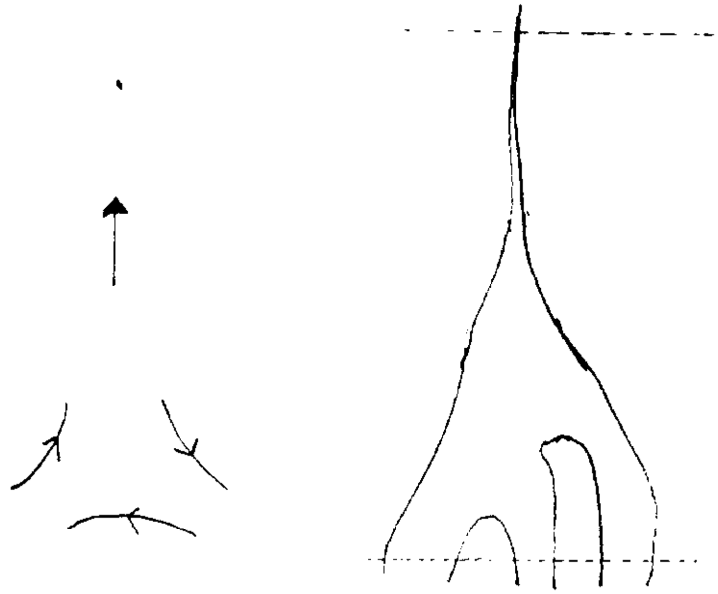


Fig. 3.

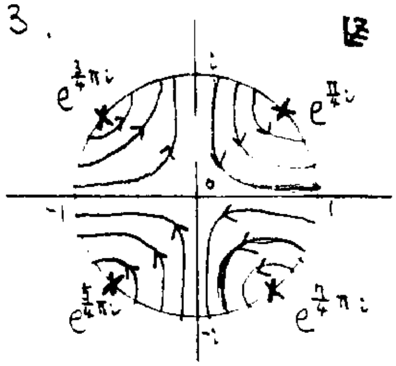


Fig. 4.

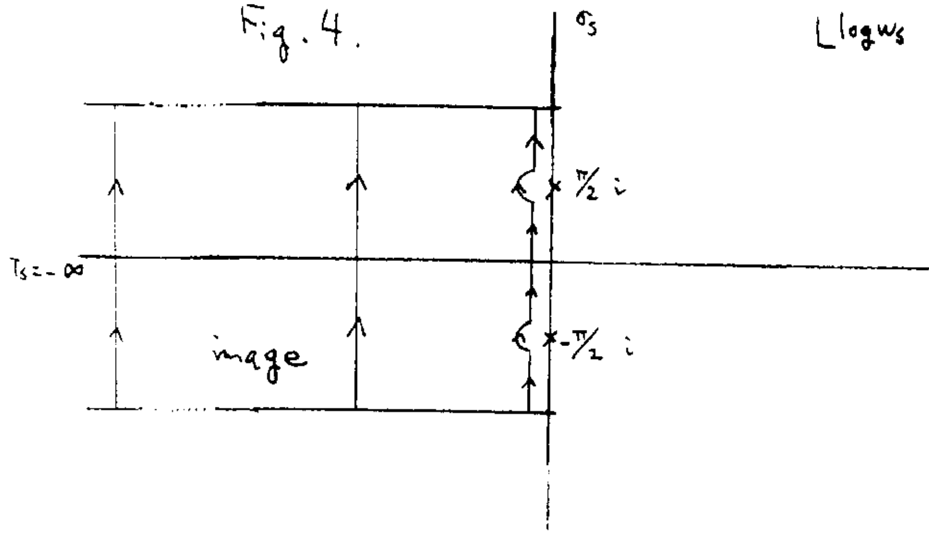


Fig. 5.

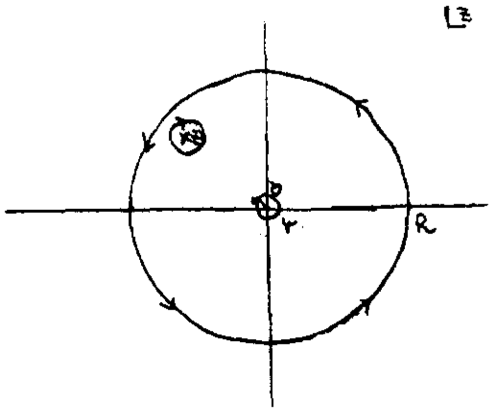


Fig. 6.

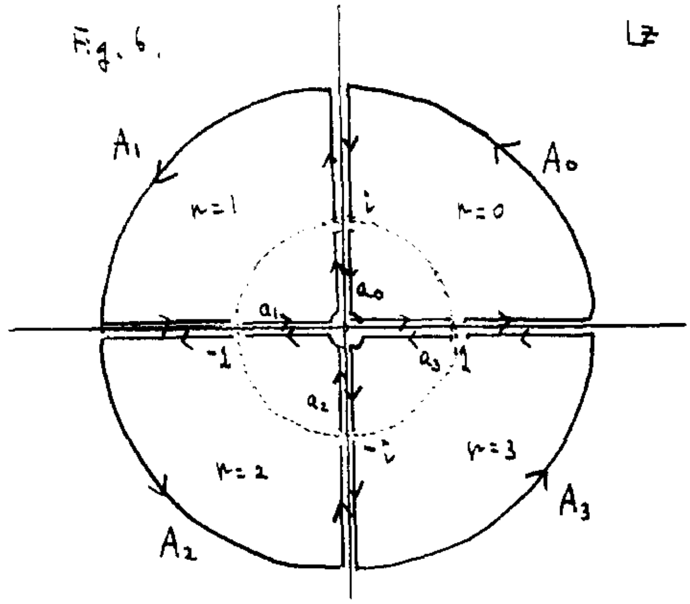


Fig. 7.

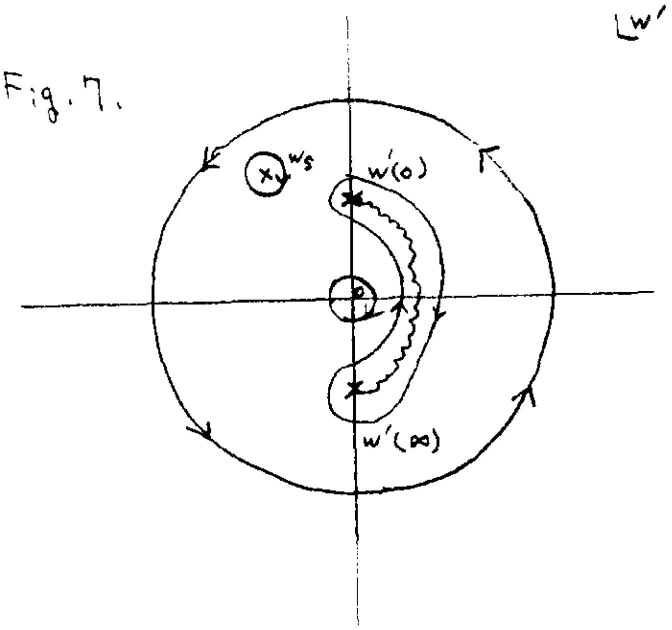


Fig. 8

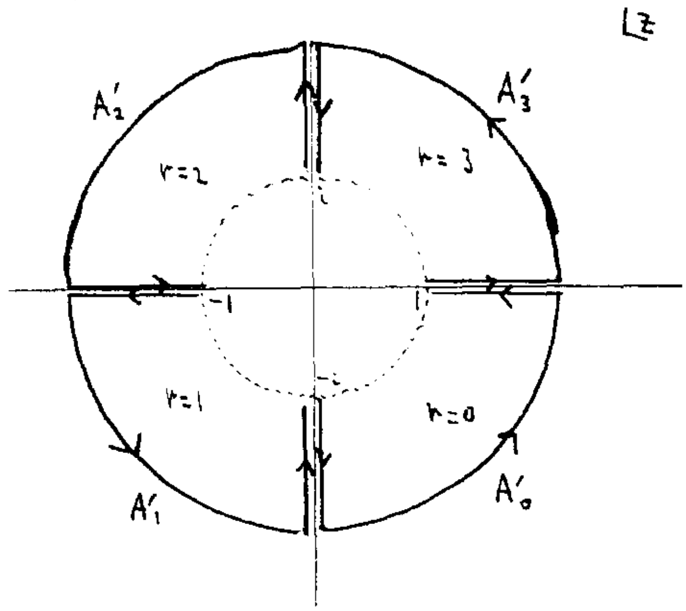


Fig. 9.

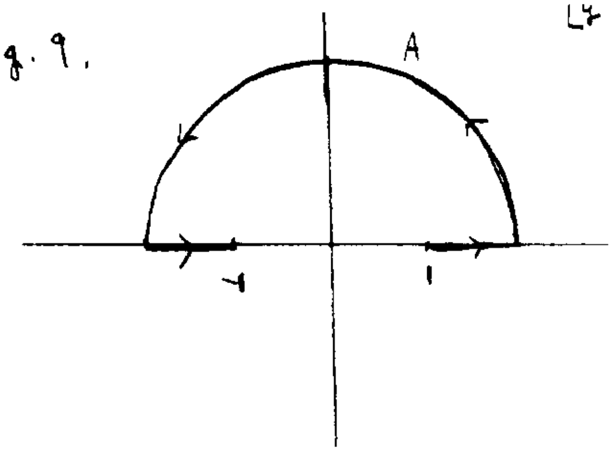


Fig. 10

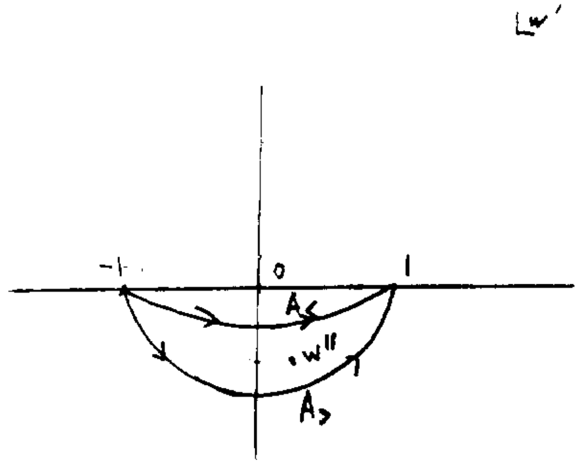
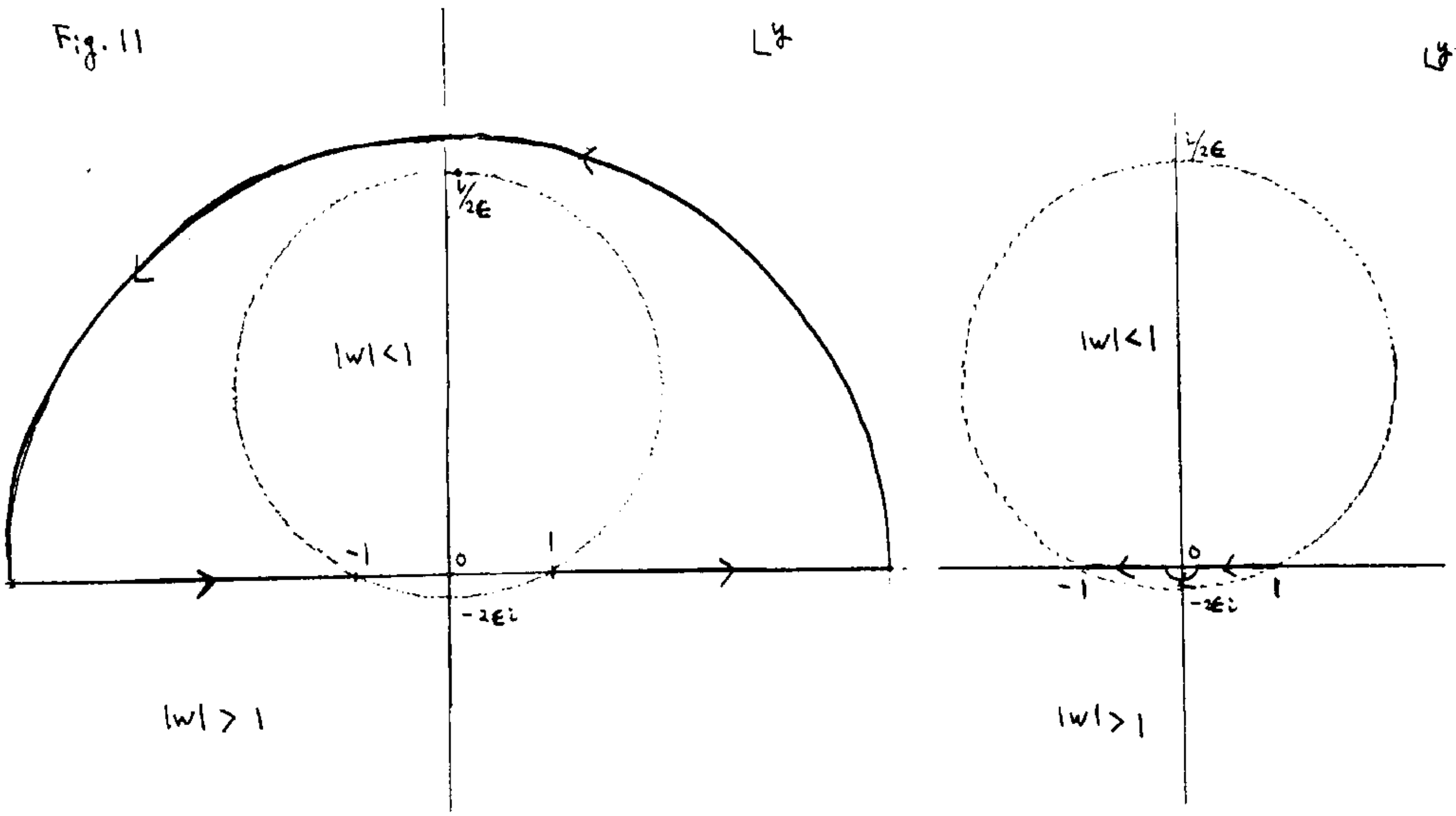


Fig. 11



$\ell \geq 2$ $\qquad\qquad$ $\ell \leq 1$

Fig. 2.